Strong Convergence of Generalized Resolvents

Lina Aryati
Department of Mathematics, Gadjah Mada University
Yogyakarta, Indonesia.
Email: lina@ugm.ac.id

ABSTRACT

Let M and $\{L_n\}$ be a linear, closed, densely defined operator and a sequence of linear, closed, densely defined operators in a Banach Space X respectively. We consider a sequence of generalized resolvents $\{R_n(\lambda)\}$, where $R_n(\lambda) = (L_n - \lambda M)^{-1}M$. In this paper, we will prove that the sequence $\{R_n(\lambda)\}$ is uniformly bounded in n and λ in any compact subset of a certain open set. Then we will concern with consideration on strong convergence of $\{R_n(\lambda)\}$. Finally we will give a criterion for the sequence $\{R_n(\lambda)\}$ converges strongly.

Keywords : *generalized resolvent, uniformly bounded, strong convergence*

1. INTRODUCTION

Resolvent plays an important role in non degenerate perturbation theory for the eigenvalue problems of linear operators. When we consider an analytic perturbation, the continuity in the norm of the resolvent in the parameter plays the fundamental role (see e.g. in Kato (1995) and Baumgärtel (1984)).

When we consider some applications, it is necessary to develop degenerate perturbation theory, instead of the non degenerate one. By defining generalized resolvents, instead of the resolvent, Aryati (2000) developed degenerate perturbation theory for degenerate eigenvalue problems. This theory has applications in quantum mechanics when we want to analyze the behavior of the Dirac equation in the non relativistic limit. As mentioned in Thaller (1992) this investigation is motivated by a practical reason; that in many cases, it is useful to replace the Dirac theory by the simpler Schrödinger theory with some corrections. Later, Aryati (2002) derived analytic perturbation of a semi simple generalized eigenvalue, and then Aryati (2004) gave a result on analytic perturbation of a discrete generalized eigenvalue.

In the non degenerate analytic perturbation, the analyticity of the resolvent is an important condition for eigenvalues to be analytic. In the case when the resolvent is not analytic, it is still possible to deduce asymptotic expansions, provided that the eigenvalues are stable (see Kato (1995)). As in analytic perturbation, the behavior of the resolvent is a basic for asymptotic perturbation theory. The stability of eigenvalues depend on the region of strong convergence for the resolvents.

In this paper, we will generalize the result given by Kato (1995) and will concern with consideration on strong convergence of generalized resolvents. Here, we deal with generalized resolvents which are different from those given by Alexandre (1999), Kimura (2005), and Verma (2007).

2. MAIN RESULTS

Let X be a Banach Space, L and M be linear, closed, densely defined operators in X. The generalized resolvent set (of L with respect to M), written by $\rho_M^{(R)}(L)$, is the set of all scalar $\lambda \in \mathbb{C}$ such that the operator $R(\lambda) = (L - \lambda M)^{-1}M$ is bounded and defined on X. Let $B(X)$ be the set of all bounded operators on X. The operator $R : \rho_M^{(R)}(L) \longrightarrow B(X)$ such that

$$\lambda \in \rho_M^{(R)}(L) \longmapsto R(\lambda) = (L - \lambda M)^{-1} M \in B(X)$$

is called the *generalized resolvent* (of L with respect to M). This definition is different from that suggested by Kato (1995), that is $(L - \lambda M)^{-1}$, and motivated by reasons that in infinite dimensional spaces, it can happen that $R(\lambda)$ is bounded, defined on X, even when $(L - \lambda M)^{-1}$ is unbounded, and in certain condition, we still have the basic properties which are similar to those in the non degenerate eigenvalue problem. Note that the generalized resolvent set can be empty (see Aryati (2000)), therefore in this paper we assume that the generalized resolvent set is not empty.

Throughout the following theorems, let $\{L_n\}$ be a sequence of linear, closed, densely defined operators in a Banach Space X. Let us define $\mathcal{D}_n = \mathcal{D}(L_n) \cap \mathcal{D}(M)$ and $(L_n - \lambda M)^{-1} : \operatorname{range} M \to \mathcal{D}_n$. Then we consider a sequence of generalized resolvents $\{R_n(\lambda)\}$, where $R_n(\lambda) = (L_n - \lambda M)^{-1} M$. We refer to Kato (1995) for the concepts of uniformly bounded, strong convergence, etc. We use the notation $s - \lim T_n = T$, for T is the strong limit of the sequence $\{T_n\}$.

Theorem 1 *Let N be a natural number. Let Δ_b be the set of all $\lambda \in \mathbb{C}$ such that M is defined on X, for every $n \geq N$, $(L_n - \lambda M)^{-1}$ are bounded, and the sequence $\{\|R_n(\lambda)\|\}_{n \geq N}$ is bounded. Let $\lambda_0 \in \Delta_b$. Then the sequence $\{R_n(\lambda)\}$ is uniformly bounded in n and λ in any compact subset Γ of $\bigcup(\lambda_0)$, where $\bigcup(\lambda_0) = \{\lambda \in \Delta_b \mid |\lambda - \lambda_0| < \|R_n(\lambda_0)\|^{-1}\}$. Furthermore Δ_b is an open set in the complex plane.*

Proof. It is easy to show that for every $\lambda, \mu \in \Delta_b$,

$$R_n(\lambda) - R_n(\mu) = (\lambda - \mu) R_n(\lambda) R_n(\mu).$$

Let $\lambda_0 \in \Delta_b$, then for every $\lambda \in \Delta_b$,

$$R_n(\lambda)\big(1 - (\lambda - \lambda_0) R_n(\lambda_0)\big) = R_n(\lambda_0). \quad (1)$$

Since for every $n \geq N$, $R_n(\lambda_0) = (L_n - \lambda_0 M)^{-1} M$ has a finite norm, we can define an open disk $\bigcup(\lambda_0) = \{\lambda \in \Delta_b \mid |\lambda - \lambda_0| < \|R_n(\lambda_0)\|^{-1}\}$. Then from (1) for every $n \geq N$ and for every $\lambda \in \bigcup(\lambda_0)$, we have the first Neumann series

$$\begin{aligned} R_n(\lambda) &= R_n(\lambda_0)\big(1 - (\lambda - \lambda_0) R_n(\lambda_0)\big)^{-1} \\ &= \sum_{k=0}^{\infty} (\lambda - \lambda_0)^k R_n(\lambda_0)^{k+1}. \end{aligned}$$

If $\|R_n(\lambda_0)\| \leq C_0$, with $C_0 > 0$, it follows that for λ with $|\lambda - \lambda_0| < \frac{1}{C_0}$, $\|R_n(\lambda)\| \leq C_0(1 - C_0|\lambda - \lambda_0|)^{-1}$. Therefore the sequence $\{R_n(\lambda)\}$ is uniformly bounded in n and λ in any compact subset Γ of $\bigcup(\lambda_0)$.

Next, we will show that Δ_b is open. Let $n \geq N$. Since M is defined on X (hence bounded) and for any $\lambda_0 \in \Delta_b$, $G_n(\lambda_0) = (L_n - \lambda_0 M)^{-1} |_{\operatorname{range} M}$ exists and has a finite norm, then in the open disk $\{\lambda \in \mathbb{C} \mid |\lambda - \lambda_0| < \|M G_n(\lambda_0)\|^{-1}\}$, we can define

$$\hat{G}_n(\lambda) = \sum_{k=0}^{\infty} (\lambda - \lambda_0)^k (G_n(\lambda_0) M)^k G_n(\lambda_0),$$

with $\hat{G}_n(\lambda_0) = G_n(\lambda_0)$. If we can show that $\hat{G}_n(\lambda) = (L_n - \lambda M)^{-1} |_{\operatorname{range} M}$ for λ in the open disk, then the proof is complete.

a). $\hat{G}_n(\lambda)(L_n - \lambda M) = 1$, on $\mathcal{D}(L_n)$. For any $u \in \mathcal{D}(L_n)$,

$$\begin{aligned} &\hat{G}_n(\lambda)(L_n - \lambda M)u = \\ &\Big(\textstyle\sum_{k=0}^{\infty} (\lambda - \lambda_0)^k (G_n(\lambda_0) M)^k G_n(\lambda_0)(L_n - \lambda M)\Big) u \\ &= \Big(\textstyle\sum_{k=0}^{\infty} (\lambda - \lambda_0)^k (G_n(\lambda_0) M)^k\Big) u \\ &- \Big(\textstyle\sum_{k=0}^{\infty} (\lambda - \lambda_0)^{k+1} (G_n(\lambda_0) M)^{k+1}\Big) u \\ &= (\lambda - \lambda_0)^0 (G_n(\lambda_0) M)^0 u = u, \end{aligned}$$

since $G_n(\lambda_0)(L_n - \lambda M)u = G_n(\lambda_0)(L_n - \lambda_0 M + \lambda_0 M - \lambda M)u = (1 - (\lambda - \lambda_0) G_n(\lambda_0) M)u$.

b). $(L_n - \lambda M)\hat{G}_n(\lambda) = 1$, on $\operatorname{range} M$. For any $u \in \operatorname{range} M$, if we define a sequence

$$\Psi_K := \Big(\sum_{k=0}^{K} (\lambda - \lambda_0)^k (G_n(\lambda_0) M)^k G_n(\lambda_0)\Big) u,$$

then $\Psi_K \in \mathcal{D}(L_n)$ and $\Psi_K \overset{K \to \infty}{\longrightarrow} \Psi = \hat{G}_n(\lambda) u$.

Moreover,

$$\begin{aligned}&(L_n-\lambda M)\Psi_K\\&=(L_n-\lambda M)\\&\quad\Big(\sum_{k=0}^{K}(\lambda-\lambda_0)^k(G_n(\lambda_0)M)^kG_n(\lambda_0)\Big)u\\&=(L_n-\lambda M)G_n(\lambda_0)\\&\quad\Big(\sum_{k=0}^{K}(\lambda-\lambda_0)^k(G_n(\lambda_0)M)^k\Big)u\\&=(1-(\lambda-\lambda_0)MG_n(\lambda_0))\\&\quad\Big(\sum_{k=0}^{K}(\lambda-\lambda_0)^k(G_n(\lambda_0)M)^k\Big)u.\end{aligned}$$

Because

$$\begin{aligned}&\|(\lambda-\lambda_0)^{K+1}(MG_n(\lambda_0))^{K+1}\|\\&\quad\le|(\lambda-\lambda_0)^{K+1}|\,\|(MG_n(\lambda_0))^{K+1}\|<1,\end{aligned}$$

we know that $\|(\lambda-\lambda_0)^{K+1}(MG_n(\lambda_0))^{K+1}\| \overset{K\to\infty}{\longrightarrow} 0$ and $(L_n-\lambda M)\Psi_K \overset{K\to\infty}{\longrightarrow} u$. The operator $(L_n-\lambda M)$ is closed, $\{\Psi_K\}$ and $\{(L_n-\lambda M)\Psi_K\}$ are convergent, hence we find that

$$\lim_{K\to\infty}\Psi_K=\Psi$$

and

$$\begin{aligned}u&=\lim_{K\to\infty}(L_n-\lambda M)\Psi_K\\&=(L_n-\lambda M)\lim_{K\to\infty}\Psi_K\\&=(L_n-\lambda M)\Psi=(L_n-\lambda M)\hat{G}_n(\lambda)u.\end{aligned}$$

Hence Δ_b is open. ∎

Let Δ_s be the set of all $\lambda\in\mathbb{C}$ such that $s-\lim R_n(\lambda)$ exists.

Theorem 2 *The set Δ_s is relatively open and closed in Δ_b. There exists $\hat{R}(\lambda)$ such that the strong convergence $R_n(\lambda)\to\hat{R}(\lambda)$ is uniform in each compact subset of Δ_s in the sense that $\|R_n(\lambda)u-\hat{R}(\lambda)u\|\to 0$ uniformly in λ in each compact subset of Δ_s for each fix $u\in X$.*

Proof. First we will show the last assertion and that Δ_s is relatively open in Δ_b. Let $\lambda_0\in\Delta_s$, $s-\lim R_n(\lambda_0)=\hat{R}(\lambda_0)$, then $s-\lim R_n(\lambda_0)^k=\hat{R}(\lambda_0)^k, k=1,2,\ldots.$ Note that for every

$$\lambda\in\{\lambda\in\Delta_b\,\big|\,|\lambda-\lambda_0|<\|R_n(\lambda_0)\|^{-1}\},$$

$$\begin{aligned}\|R_n(\lambda)\|&=\|\sum_{k=0}^{\infty}(\lambda-\lambda_0)^kR_n(\lambda_0)^{k+1}\|\\&\le\sum_{k=0}^{\infty}|(\lambda-\lambda_0)|^k\|R_n(\lambda_0)^{k+1}\|.\end{aligned}$$

If $\|R_n(\lambda_0)\|\le C_0$, then

$$\|R_n(\lambda)\|\le\sum_{k=0}^{\infty}|(\lambda-\lambda_0)|^k{C_0}^{k+1}.$$

It follows that $s-\lim R_n(\lambda)$ exists and for λ with $|\lambda-\lambda_0|<\frac{1}{C_0}$,

$$\begin{aligned}s-\lim R_n(\lambda)&=\sum_{k=0}^{\infty}(\lambda-\lambda_0)^k\hat{R}(\lambda_0)^{k+1}\\&=R(\lambda_0)(1-(\lambda-\lambda_0)\hat{R}(\lambda_0))^{-1}\\&=\hat{R}(\lambda).\end{aligned}$$

Next we will show that Δ_s is relatively closed in Δ_b. Let $\lambda\in\Delta_b$ and assume that in each neighborhood of λ there is a $\lambda_0\in\Delta_s$. Since $\lambda\in\Delta_b$, then there exists $C>0$ such that $\|R_n(\lambda)\|\le C$. Take $\lambda_0\in\Delta_s$ with $|\lambda-\lambda_0|<\frac{1}{2C}$. Then $\|R_n(\lambda_0)\|\le 2C=C_0$, and $s-\lim R_n(\lambda)$ exists, since $|\lambda-\lambda_0|<\frac{1}{2C}=\frac{1}{C_0}$. Hence $\lambda\in\Delta_s$. ∎

Note that for each $\lambda,\mu\in\Delta_s$, the strong limit $\hat{R}(\lambda)$ and $\hat{R}(\mu)$ satisfies the first resolvent equation:

$$\hat{R}(\lambda)-\hat{R}(\mu)=(\lambda-\mu)\hat{R}(\lambda)\hat{R}(\mu),\qquad(2)$$

hence the strong limit is a pseudoresolvent. Furthermore, it is easy to prove that $\hat{R}(\lambda)$ and $\hat{R}(\mu)$ commute.

Let $\lambda,\mu\in\Delta_s$. If $\hat{R}(\mu)=0$, then for every $u\in X$ from (2) we get

$$\begin{aligned}\hat{R}(\lambda)u&=\big((\hat{R}(\mu)+(\lambda-\mu)\hat{R}(\lambda)\hat{R}(\mu)\big)u\\&=\hat{R}(\mu)u+(\lambda-\mu)\hat{R}(\lambda)\hat{R}(\mu)u=0,\end{aligned}$$

therefore $\ker(\hat{R}(\lambda))$ is independent of λ. Let $u\in\text{range}(\hat{R}(\mu))$, then $u=\hat{R}(\mu)v$, for some $v\in X$. The first resolvent equation (2) gives:

$$\begin{aligned}u=\hat{R}(\mu)v&=\hat{R}(\lambda)v-(\lambda-\mu)\hat{R}(\lambda)\hat{R}(\mu)v\\&=\hat{R}(\lambda)v-(\lambda-\mu)\hat{R}(\lambda)u\\&=\hat{R}(\lambda)\big(v-(\lambda-\mu)u\big)\\&=\hat{R}(\lambda)w,\end{aligned}$$

where $w=v-(\lambda-\mu)u$. Hence $\text{range}(\hat{R}(\lambda))$ is also independent of λ.

If it is provided certain conditions, then the strong limit becomes a generalized resolvent.

Theorem 3 *Let Δ_s be nonempty and M be defined every where. If for every $\lambda \in \Delta_s$ there exists $\hat{G}(\lambda)$ such that*

$$\begin{aligned} & s-\lim(L_n-\lambda M)^{-1}=\hat{G}(\lambda), \\ & \ker(\hat{G}(\lambda)|_{\text{range } M})=\{0\} \\ \textit{and} \quad & \ker(M\hat{G}(\lambda)|_{\text{range } M})=\{0\}, \end{aligned}$$

then there exists a unique operator L such that $\hat{R}(\lambda)=\hat{G}(\lambda)M=(L-\lambda M)^{-1}M=R(\lambda)$ is a generalized resolvent. Moreover $\Delta_s = \rho_M^{(R)}(L)\cap\Delta_b$.

Proof. For every $u \in \text{range}(\hat{G}(\lambda)|_{\text{range } M})$, u can be written as $u=\hat{G}(\lambda)Mv(\lambda)$. Since $\hat{R}(\lambda)$ and $\hat{R}(\mu)$ commute, then for every $\lambda,\mu \in \Delta_s$,

$$\begin{aligned} & \hat{G}(\lambda)M\hat{G}(\mu)M(v(\mu)-v(\lambda)) \\ & =\hat{R}(\lambda)\hat{R}(\mu)(v(\mu)-v(\lambda)) \\ & =\hat{R}(\lambda)\hat{G}(\mu)Mv(\mu)-\hat{R}(\mu)\hat{G}(\lambda)Mv(\lambda) \\ & =\big(\hat{R}(\lambda)-\hat{R}(\mu)\big)u \\ & =\hat{R}(\lambda)\hat{R}(\mu)(\lambda-\mu)u \\ & =\hat{G}(\lambda)M\hat{G}(\mu)M(\lambda-\mu)u. \end{aligned}$$

Let $\lambda \in \Delta_s$. Since it is provided that $\ker(\hat{G}(\lambda)|_{\text{range } M})=\{0\}$ and

$$\ker(M\hat{G}(\lambda)|_{\text{range } M})=\{0\},$$

therefore $M(v(\mu)-v(\lambda)) = M(\lambda-\mu)u$ or $Mv(\mu)+M(\mu)u=v(\lambda)+M(\lambda)u$, and hence $Mv(\lambda)+M(\lambda)u$ is independent of λ. Define an operator L with $Lu=Mv(\lambda)+M(\lambda)u$, then L is a linear operator in X with

$$\mathcal{D}(L)=range(\hat{G}(\lambda)|_{\text{range } M})$$

and $(L-\lambda M)u=Mv(\lambda)=\big(\hat{G}(\lambda)\big)^{-1}u$, for every $u \in range(\hat{G}(\lambda)|_{\text{range } M})$. Hence $\hat{G}(\lambda)=(L-\lambda M)^{-1}$ and

$$\hat{R}(\lambda)=\hat{G}(\lambda)M=(L-\lambda M)^{-1}M=R(\lambda)$$

is a generalized resolvent. Therefore $\Delta_s \subset \rho_M^{(R)}(L)$, and hence $\Delta_s \subset \rho_M^{(R)}(L)\cap\Delta_b$. Next, let $\lambda,\lambda_0 \in \rho_M^{(R)}(L)\cap\Delta_b$. It follows from the first resolvent equation (2) that

$$\begin{aligned} R_n(\lambda)-R(\lambda)=\big(1+(\lambda-\lambda_0)R_n(\lambda)\big)\big(R_n(\lambda_0) \\ -R(\lambda_0)\big)\big(1+(\lambda-\lambda_0)R(\lambda)\big). \qquad (3) \end{aligned}$$

For $\lambda_0 \in \Delta_s$ such that there exists $\hat{G}(\lambda_0)$ such that $s-\lim(L_n-\lambda_0 M)^{-1}=\hat{G}(\lambda_0)$, $\ker(\hat{G}(\lambda_0)|_{\text{range } M})=\{0\}$ and $\ker(M\hat{G}(\lambda_0)|_{\text{range } M})=\{0\}$, we have

$$s-\lim R_n(\lambda_0)=\hat{R}(\lambda_0)=R(\lambda_0).$$

From Theorem 1 we know that $\{R_n(\lambda)\}$ is uniformly bounded, therefore (3) gives $s-\lim R_n(\lambda)=R(\lambda)$, hence $\lambda \in \Delta_s$ or $\rho_M^{(R)}(L)\cap\Delta_b \subset \Delta_s$. ∎

3. CONCLUDING REMARK

It is proved that under certain conditions, the sequence of generalized resolvents converges strongly. This result on the strong convergence of generalized resolvents gives a possibility to continue research in development of asymptotic degenerate perturbation theory for degenerate eigenvalue problems.

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