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$(m,n)-$Closed and Quasi $(m,n)-$Closed Ideals

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Abstract. Throughout this paper, all rings considered are commutative rings R with identity 1_R. Let m and n be natural numbers such that $1 \leq n < m$. A proper ideal I of R is called an $(m,n)-$closed ideal if for every $x \in R$ with $x^m \in I$ implies $x^n \in I$. An $(m,n)-$closed ideal generalizes semi $n-$absorbing ideal and, hence, also generalizes semiprime ideal. A proper ideal I of R is called a quasi $(m,n)-$closed ideal if for every $x \in R$ with $x^m \in I$ implies $x^n \in I$ or $x^{m-n} \in I$. Therefore, a quasi $(m,n)-$closed generalizes an $(m,n)-$closed ideal. Research related to these ideals is referred to Anderson and Badawi (2017) and Khashan and Celikel (2024). In this paper, the authors presented several new properties related to these ideals that are not discussed in the two main references.

Keywords: $(m,n)-$closed ideal, quasi $(m,n)-$closed ideal, semiprime ideal, commutative ring with identity.

1. INTRODUCTION

The idea of ideal comes from the last Fermat theorem. Its proof became an open problem until Andrew Wiles found it in 1994. Before Andrew Wiles, Ernst Kummer (1810-1893) tried to solve it. During his work, he introduced a concept of "ideal number". Motivated from his idea, Richard Dedekind (1831-1916) invented ideal concept in ring theory [1].

Prime ideal is one of many concepts in ring theory. It is motivated from the prime concept in number theory. Generalization of prime ideal has been found by many mathematician. For example, Anderson and Badawi [2] define an $n-$absorbing ideal as a generalization of prime ideal and conclude that a prime ideal is just $1-$absorbing ideal. This ideal has been used by Choi [3] to prove Anderson and Badawi's conjectures in locally divided commutative rings. On the other

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hand, Moghimi and Naghani [4] define the $n-$Krull dimension of commutative ring R as a supremum of the lengths of chains of $n-$absorbing ideals of R.

There is another kind of ideal in ring theory. It is called a semiprime or radical ideal. Generally, a prime ideal is obviuos a semiprime ideal. However, its converse need not to be true. Not only prime ideal, Anderson and Badawi [**?**] define a semi $n-$absorbing ideal as a generalization of semiprime ideal. They conclude that a semiprime ideal is just a semi $1-$absorbing ideal. They also prove that every $n-$absorbing ideal is semi $n-$absorbing, but its converse need not to be true in general.

Anderson and Badawi [5] also generalize semi $n-$absorbing ideal. They define it as $(m, n)-$closed ideal for some natural numbers m and n. Therefore, an $(m, n)-$closed ideal also generalizes semiprime ideal. As a result of their research, one of its basic properties is every semiprime ideals are $(m, n)-$closed ideal for every natural numbers m and n.

Ideal generated by 2^{m-2}, i.e. $\langle 2^{m-2}\rangle$, is not $(m, 2)-$closed ideal in $\mathbb{Z}$ for every natural numbers $m \geq 5$. Khashan and Celikel [6] define a generalization of $(m, n)-$closed ideal as quasi $(m, n)-$closed ideal. By defining this ideal, ideal $\langle 2^{m-2}\rangle$ is a quasi $(m, 2)-$closed ideal in $\mathbb{Z}$ for every natural numbers $m \geq 5$. In addition, they also stated a sufficient condition for quasi $(m, n)-$closed ideal being $(m, n)-$closed ideal.

Many properties of $(m, n)-$closed ideal and quasi $(m, n)-$closed has been produced by [5] and [6] respectively. On the other hand, the applications of $(m, n)-$closed to several kind of rings also have been found by several papers. Issoual *et al.* [7] have investigated $(m, n)-$closed ideal related to the amalgamated ring $A \bowtie^f J$ for (A, B) be a pair of rings, $f : A \to B$ be a ring homomorphism and J be an ideal of B. Badawi *et al.* [8] also have explored $(m, n)-$closed ideal related to trivial ring extension $R = A(+)M$ for A is a commutative ring and M is an $A-$module. In this paper, the authors give several new properties related to these ideals that are not discussed in [5] and [6].

2. SOME CONCEPTS ABOUT $(m, n)-$CLOSED AND QUASI $(m, n)-$CLOSED IDEALS

In this section, we briefly give some explanations about $(m, n)-$closed and quasi $(m, n)-$closed ideal. Throughout this paper, all rings considered are commutative rings R with identity 1_R and all ring homomorphisms preserve the identity.

Let R be a commutative ring with identity. An ideal is proper if $I \neq R$. For some proper ideal I of R, the radical of I, denoted by $\sqrt{I}$, is defined by $\{x \in R | x^n \in I \text{ for some } n \in \mathbb{N}\}$. An element of R is called a nilpotent element if there exists a natural number n such that $x^n = 0_R$. The set of all nilpotent elements is called nilradical of R, denoted by $\text{nil}(R)$. Clearly, we have $\sqrt{\{0_R\}} = \text{nil}(R)$.

A proper ideal I of R is called a prime ideal if whenever $xy \in I$ for every $x, y \in R$, implies $x \in I$ or $y \in I$ [9]. Ideal generated by a prime number p of $\mathbb{Z}$, i.e. $\langle p \rangle$, is an example of prime ideal. A semiprime ideal is a proper ideal with the property that whenever $x^2 \in I$ for every $x \in R$, implies $x \in I$ [9]. Every prime ideal is semiprime ideal, but its converse need not to be true. Ideal generated by $\langle 6 \rangle$ of $\mathbb{Z}$, i.e. $\langle 6 \rangle$, is a semiprime ideal of $\mathbb{Z}$ but it is not a prime ideal of $\mathbb{Z}$. Anderson and Badawi [2] generalize a prime ideal to $n-$absorbing ideal.

Definition 2.1. [2] *Let n be a natural number. A proper ideal I of R is called an $n-$absorbing ideal of R if whenever $x_1x_2 \dots x_nx_{n+1} \in I$ for every $x_1, x_2, \dots, x_n, x_{n+1} \in I$, then there are n of x_i's whose product is in I.*

In particular, an $n-$absorbing ideal is expanded from $2-$absorbing ideal concept that has been explored by Badawi [10] and Payrovi and Babaei [**?**]. Same with prime ideal, Anderson and Badawi [5] generalize a semiprime ideal to semi $n-$absorbing ideal.

Definition 2.2. [5] *Let n be a natural number. A proper ideal I of R is called a semi $n-$absorbing ideal if whenever $x^{n+1} \in I$ for every $x \in R$, implies $x^n \in I$.*

Thus, an $n-$absorbing ideal is a semi $n-$absorbing ideal. It follows from definitions, a prime ideal and a semiprime ideal are a $1-$absorbing ideal and a semi $n-$absorbing ideal, respectively.

Anderson and Badawi [5] also generalize a semi $n-$absorbing to an $(m, n)-$closed ideal.

Definition 2.3. [5] *Let m and n be natural numbers such that $1 \leq n < m$. A proper ideal I of R is called an $(m, n)-$closed ideal if whenever $x^m \in I$ for every $x \in R$, implies $x^n \in I$.*

Note that Mostafanasab and Darani [11] define a proper ideal I of R to be a semi $(m, n)-$absorbing ideal if I is an $(m, n)-$closed ideal. Here is an example of an $(m, n)-$closed ideal.

Example 2.4. *Let $R = \mathbb{Z}[x, y]$ and $I = \langle x^2, 2xy, y^2 \rangle$. We will prove that I is an $(m, 2)-$closed ideal of R for any natural numbers $m \geq 3$. Suppose $f(x, y) \in R$ such that $(f(x, y))^m \in I$. We have $f(x, y) \in \sqrt{I} = \langle x, y \rangle$. Then, $f(x, y)$ can be written as $f(x, y) = g(x, y)x + h(x, y)y$ for some $g(x, y), h(x, y) \in R$. By quadrating both sides, we get $(f(x, y))^2 = (g(x, y))^2x^2 + 2xyg(x, y)h(x, y) + (h(x, y))^2y^2 \in I$.*

By induction, we can prove that a semiprime ideal is an $(m, n)-$closed ideal for every natural numbers m and n. Note that an $(m, n)-$closed ideal is also an $(m', n')-$closed ideal for every natural numbers $m' \leq m$ and $n' \geq n$. The complete properties of an $(m, n)-$closed ideal can be found on [5].

The concept of $(m, n)-$closed ideal is generalized by Khashan and Celikel [6]. They define it as a quasi $(m, n)-$closed ideal.

Definition 2.5. [6] *Let m and n be natural numbers such that $1 \leq n < m$. A proper ideal I of R is called a quasi $(m, n)-$closed ideal if whenever $x^m \in I$ for every $x \in R$, implies $x^n \in I$ or $x^{m-n} \in I$.*

Here is an example of a quasi $(m, n)-$closed ideal.

Example 2.6. *Let* $R = \mathbb{Z}$*. We will show that* $I = \langle 2^{m-2} \rangle$ *is a quasi* $(m, 2)-$*closed ideal of* R *for every natural numbers* $m \geq 5$*. Let* $x \in R$ *such that* $x^m \in I$*. By the definition of* I*, we have* $2^{m-2} | x^m$*. This implies that* $2|x$*. Consequently, we get* $x^{m-2} \in I$*. However,* I *is not an* $(m, 2)-$*closed ideal since* $2^m \in I$ *but* $2^2 \notin I$*.*

It is clear that a proper ideal I of R is a quasi (m, n)- closed ideal if and only if I is an $(m, n)-$closed or an $(m, m-n)-$closed ideal. Furthermore, a proper ideal I of R is a quasi $(m, n)-$closed ideal if and only if I is a quasi $(m, m-n)-$closed ideal. The comprehensive properties about quasi $(m, n)-$closed ideal can be found on [6].

3. RESULT AND DISCUSSION

In this section, we deliver several new properties related to $(m, n)-$closed ideals and quasi $(m, n)-$closed ring that are not discussed in [5] and [6].

3.1. **$(m, n)-$closed Ideal.** Let m and n be natural numbers such that $1 \leq n < m$. Corollary 2.4 in [5] shows that if $I_1, I_2, \ldots, I_k$ are $(m, n)-$closed ideals of a commutative ring R with identity 1_R, then the intersection $I_1 \cap I_2 \cap \cdots \cap I_k$ is also an $(m, n)-$closed ideal of R. Fortunately, this fact can be also extended to the collection of $(m, n)-$closed ideals of R.

Theorem 3.1. *Let* R *be a commutative ring with identity* 1_R *and* m, n *be natural numbers such that* $1 \leq n < m$*. If* $\{I_\alpha | \alpha \in \Lambda\}$*, where* $\emptyset \neq \Lambda$ *denotes an indexing set, is a collection of* $(m, n)-$*closed ideals of* R*, then* $\cap_{\alpha \in \Lambda} I_\alpha$ *is an* $(m, n)-$*closed ideal of* R*.*

Proof. Let $x \in R$ such that $x^m \in \cap_{\alpha \in \Lambda} I_\alpha$, then $x^m \in I_\alpha$ for every $\alpha \in \Lambda$. Since I_α is an $(m, n)-$closed ideal for every $\alpha \in \Lambda$, we have $x^n \in \cap_{\alpha \in \Lambda}$. □

In commutative ring R, the set nil(R) is an ideal of R. Its proof can be found on [12]. For an ideal I of R, the set $\sqrt{I}$ is also an ideal of R [13]. Now, we can prove that nil(R) and $\sqrt{I}$ are $(m, n)-$closed ideals for every natural numbers m and n such that $1 \leq n < m$.

Theorem 3.2. *If* R *is a commutative ring with identity* 1_R*, then* nil(R) *and* $\sqrt{I}$*, for an ideal* I *of* R*, are* $(m, n)-$*closed ideals of* R *for every natural numbers* m *and* n*.*

Proof. It is enough to show that nil(R) and $\sqrt{I}$ are semiprime ideals of R. Let $x \in R$ such that $x^2 \in \sqrt{I}$. Based on the definition of $\sqrt{I}$, there is a natural number k such that $x^{2k} = (x^2)^k \in I$. Since $2k$ is also a natural number, we get $x \in \sqrt{I}$. Now, let $y \in R$ such that $y^2 \in$ nil(R). By definition, there is a natural number j such that $y^{2j} = (y^2)^j = 0_R$. Moreover, we get $y \in$ nil(R). □

It is easy to verify that if I is a proper ideal of a commutative ring R with identity 1_R, then $I[x]$ is a proper ideal of a polynomial ring $R[x]$. It is also obvious that $I = I[x] \cap R$. Hence, we get a fact below. Definitely, this fact is a consequence of the first statement of Corollary 2.11 on [5].

Corollary 3.3. *Let R be a commutative ring with identity 1_R and m, n be natural numbers such that $1 \leq n < m$. If $I[x]$ is an $(m, n)-$closed ideal of $R[x]$, then I is an $(m, n)-$closed ideal of R.*

Proof. Note that R can be embedded to $R[x]$, so we have $R \subseteq R[x]$. Applying the first statement of Corollary 2.19 on [5] and the fact $I = I[x] \cap R$, we have I is an $(m, n)-$closed ideal of R. □

In order to prove our next results, we present the second statement of Corollary 2.11 on [5].

Corollary 3.4. [5] *Let R be a commutative ring with identity 1_R and m, n be natural numbers such that $1 \leq n < m$. If $I \subseteq J$ are proper ideals of R, then J/I is an $(m, n)-$closed ideal of R/I if and only if J is an $(m, n)-$closed ideal of R.*

Proof. Note that the mapping $f : R \to R/I$ defined by $f(r) = r + I$ is a ring epimorphism. In addition, for this epimorphism f, we also have $f(J) = J/I$ and $f^{-1}(J/I) = J$. Let $u \in \ker(f)$, then $0_R + I = f(x) = x + I$. Consequently, we have $x = x - 0_R \in I \subseteq J$. This means that $\ker(f) \subseteq J$. Finally, by applying Theorem 2.10 on [5], the proof is done. □

The first result is the addition $I_1 + I_2 + \cdots + I_p$ is $(m, n)-$closed ideal of R if $I_1, I_2, \ldots, I_p$ are $(m, n)-$closed ideals of R.

Theorem 3.5. *Let R be a commutative ring with identity 1_R and m, n be natural numbers such that $1 \leq n < m$. If $I_1, I_2, \ldots, I_p$ are $(m, n)-$closed ideals of R, then $I_1 + I_2 + \cdots + I_p$ is an $(m, n)-$closed ideal of R.*

Proof. We will prove by induction on p. Let $p = 2$, i.e. I_1 and I_2 are $(m, n)-$closed ideals of R. Define $f : I_1 \to \frac{I_1+I_2}{I_2}$ by $f(i) = i + I_2$ for every $i \in I_1$, then f is well defined. Moreover, f is a ring homomorphism from $I_1$to $\frac{I_1+I_2}{I_2}$. Let $j \in \ker(f)$, then we have $j \in I_2$. Since $j \in I_1$, we have $\ker(f) \subseteq I_1 \cap I_2$. For every $x \in I_1 \cap I_2$, we see that $x = x + I_2 = 0_R + I_2$ and moreover, $x \in \ker(f)$. This means that $I_1 \cap I_2 \subseteq \ker(f)$. Next, we also have $\mathrm{im}(f) = \frac{I_1+I_2}{I_2}$. It follows from the fundamental theorem of ring homomorphism that $\frac{I_1}{I_1 \cap I_2} \cong \frac{I_1+I_2}{I_2}$. By Corollary 3.4, $\frac{I_1}{I_1 \cap I_2}$ is an $(m, n)-$closed ideal in $R/(I_1 \cap I_2)$. Using the isomorphism that we have just proven, $\frac{I_1+I_2}{I_2}$ is an $(m, n)-$closed ideal of R/I_2. Using Corollary 3.4 again, $I_1 + I_2$ is an $(m, n)-$closed ideal of R. Assume that the theorem is true for $p = k$ and we will prove that it is also true for $p = k + 1$. Let $J = I_1 + I_2 + \cdots + I_k$. By using same method for $p = 2$ and induction hypothesis, ideal $J + I_{k+1} = I_1 + I_2 + \cap + I_k + I_{k+1}$ is an $(m, n)-$closed ideal of R. □

Not only Theorem 3.5, Corollary 3.4 is also useful to prove the following theorem.

Theorem 3.6. *Given a proper ideal I of a commutative ring R with identity 1_R and m, n are natural numbers such that $1 \leq n < m$. Ideal $I + \langle x \rangle$ is an $(m, n)-$closed ideal of $R[x]$ if and only if I is an $(m, n)-$closed ideal of R.*

Proof. It is sufficient to prove that $\frac{I+\langle x\rangle}{I} \cong I$. Define an $f : I + \langle x\rangle \to I$ by $f(i + p(x)) = i$ for every $i + p(x) \in I + \langle x\rangle$. Thus, f is well defined and is also a ring homomorphism. Let $y + h(x) \in \ker(f)$, then we have $y = f(y + p(x)) = 0_R$. Moreover, we get $y + p(x) = p(x) \in \langle x\rangle$. This means $\ker(f) \subseteq \langle x\rangle$. Let $f(x) \in \langle x\rangle$, then $r(x) = u(x)x$ for some $u(x) \in R[x]$. Applied it to f, $f(r(x)) = f(0_R + r(x)) = 0_R$. Consequently, $\langle x\rangle$ is a subset of $\ker(f)$. By the definition of f, we have $\mathrm{im}(f) = I$. It follows from the fundamental theorem of ring homomorphism that $\frac{I+\langle x\rangle}{\langle x\rangle} \cong I$. Using Corollary 3.4, $I + \langle x\rangle$ is an $(m, n)-$closed ideal of $R[x]$ if and only if $\frac{I+\langle x\rangle}{\langle x\rangle}$ is an $(m, n)-$closed ideal of $\frac{R[x]}{\langle x\rangle}$. Using the fact $\frac{I+\langle x\rangle}{\langle x\rangle} \cong I$, the proof is completely done. □

Let $\{R_i | i = 1, 2, \dots, k\}$ be a finite collection of commutative rings R_i with identity 1_{R_i}. A direct product $\prod_{i=1}^{k} R_i$ forms commutative ring with identity $(1_{R_1}, 1_{R_2}, \dots, 1_{R_k})$ under componentwise addition and multiplication operations. If I_i is a proper ideal of R_i, for every $i = 1, 2, \dots, k$, then $\prod_{i=1}^{k} I_i$ is a proper ideal of $\prod_{i=1}^{k} R_i$. Now, we extend Theorem 2.12 on [5].

Theorem 3.7. *Let R_i be commutative ring with identity 1_{R_i} for every $i = 1, 2, \dots, k$. If I_i is an $(m_i, n_i)-$closed ideal of R_i for every $i = 1, 2, \dots, k$, then $\prod_{i=1}^{k} I_i$ is an $(m, n)-$closed ideal of $\prod_{i=1}^{k} R_i$ for every natural numbers $m \le \min\{m_1, m_2, \dots, m_k\}$ and $n \ge \max\{n_1, n_2, \dots, n_k\}$.*

Proof. Let $y \in \prod_{i=1}^{k} R_i$ such that $y^m \in \prod_{i=1}^{k} I_i$. Then y^m can be expressed as $y^m = (r_1, r_2, \dots, r_k)^m = (r_1^m, r_2^m, \dots, r_k^m) \in \prod_{i=1}^{k} I_i$. This means that $r_i^m \in I_i$ for every i. Furthermore, we have $r_i^{m_i} = r_i^{m_i-m} r_i^m \in I_i$ for every i. Note that for any $i = 1, 2, \dots, k$, I_i is an $(m_i, n_i)-$closed ideal, so we get $r_i^{n_i} \in I_i$ for every $i = 1, 2, \dots, k$. Moreover, it is obvious that $r_i^n = r_i^{n-n_i} r^{n_i} \in I_i$ for every $i = 1, 2, \dots, k$. Finally, we get $y^n = (r_1^n, r_2^n, \dots, r_k^n) = (r_1, r_2, \dots, r_k)^n \in \prod_{i=1}^{k} I_i$. □

The zero ideal $\{0_R\}$ is a prime ideal of an integral domain R, since integral domain does not have zero divisor elements. Consequently, ideal $\{0_R\}$ is a semiprime ideal of an integral domain R. Moreover, we have $\{0_R\}$ is an $(m, n)-$closed ideal of an integral domain R for every natural numbers m and n such that $1 \le n < m$. Finally, we have an immediate consequnece of Theorem 2.10 on [5].

Corollary 3.8. *Let R be a commutative ring with identity 1_R and S be an integral domain. If $f : R \to S$ is a ring homomorphism, then $\ker(f)$ is an $(m, n)-$closed ideal of R for every natural numbers m and n such that $1 \le n < m$.*

Proof. Using Theorem 2.10 on [5] and $\{0_S\}$ is an $(m, n)-$closed ideal for every $1 \le n < m$, $f^{-1}(\{0_S\}) = \ker(f)$ is an $(m, n)-$closed ideal for every natural numbers m and n such that $1 \le n < m$. □

We give an example to understand Corollary 3.8.

Example 3.9. *Consider a ring homomorphism $f : \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}$ defined by $f((a,b)) = a$ for every $(a,b) \in \mathbb{Z}\times\mathbb{Z}$. Using Corollary 3.8, we get*

$$\begin{aligned}\ker(f) &= \{(a,b) \in \mathbb{Z}\times\mathbb{Z} | a = f((a,b)) = 0\} \\ &= \{(0,b) | b \in \mathbb{Z}\} = \langle (0,1)\rangle\end{aligned}$$

is an $(m,n)-$closed ideal of $\mathbb{Z}\times\mathbb{Z}$ for every natural numbers m and n such that $1 \le n < m$.

3.2. **Quasi $(m,n)-$closed Ideal.** Let m and n be natural numbers such that $1 \le n < m$ and R be a commutative ring with identity 1_R. Define three collections,

$$\begin{aligned}\mathcal{A}_{(m,n)} &= \{I | I \text{ is an } (m,n) - \text{closed ideal of } R\}, \\ \mathcal{A}'_{(m,n)} &= \{I | I \text{ is an } (m,m-n) - \text{closed ideal of } R\}, \text{ and} \\ \mathcal{B}_{(m,n)} &= \{I | I \text{ is a quasi } (m,n) - \text{closed ideal of } R\}.\end{aligned}$$

Note that $\mathcal{A}_{(m,n)} \neq \emptyset$ since nil(R) is an $(m,n)-$closed ideal. This is also true for $\mathcal{A}'_{(m,n)}$. Since a quasi $(m,n)-$closed ideal is a generalization from $(m,n)-$closed ideal and $(m,m-n)-$closed ideal, then $\mathcal{A}_{(m,n)} \subseteq \mathcal{B}_{(m,n)}$ and $\mathcal{A}'_{(m,n)} \subseteq \mathcal{B}_{(m,n)}$. It is easy to prove that $\mathcal{A}_{(m,n)} = \mathcal{B}_{(m,n)}$ if $m \le 2n$ and $\mathcal{A}'_{(m,n)} = \mathcal{B}_{(m,n)}$ if $m \ge 2n$.

From the definition of quasi $(m,n)-$closed ideal, a quasi $(m,n)-$closed ideal is a quasi $(m,m-n)-$closed ideal and vice versa. However, this fact need not to be true for $(m,n)-$closed ideal. Example 2.2 on [5] shows that $\langle 16\rangle$ is semi 2-absorbing (i.e. $(3,2)-$closed) ideal of $\mathbb{Z}$. Unfortunately, if we choose $x = 4 \in \mathbb{Z}$, then we have $4^3 \in \langle 16\rangle$ and $4 \notin \langle 16\rangle$. Hence, $\langle 16\rangle$ is not a $(3,1)-$closed ideal of $\mathbb{Z}$. Conversely, an $(m,m-n)-$closed ideal need not to be true an $(m,n)-$closed ideal. Ideal $\langle 8\rangle$ is a $(4,4-1)-$closed ideal of $\mathbb{Z}$, since for every $x \in \mathbb{Z}$ with $x^4 \in \langle 8\rangle$, then $x \in \sqrt{\langle 8\rangle} = \langle 2\rangle$ and so that $x^3 \in \langle 8\rangle$. However, there is $2 \in \mathbb{Z}$ with $2^4 \in \langle 8\rangle$ and $2 \notin \langle 8\rangle$. Hence, $\langle 8\rangle$ is not a $(4,1)-$closed ideal of $\mathbb{Z}$.

On the above paragraph, we have $\mathcal{A}_{(m,n)} = \mathcal{B}_{(m,n)}$ if $m \le 2n$ and $\mathcal{A}'_{(m,n)} = \mathcal{B}_{(m,n)}$ if $m \ge 2n$. Hence, those facts make an immediate consequence.

Corollary 3.10. *Let R be a commutative ring with identity 1_R and m,n be natural numbers such that $1 \le n < m$. If $m = 2n$, then the following statements are equivalent.*

1. *A proper ideal I of R is an $(m,n)-$closed ideal of R.*
2. *A proper ideal I of R is an $(m,m-n)-$closed ideal of R.*
3. *A proper ideal I of R is a quasi $(m,n)-$closed ideal of R.*

Recall that proper ideals I_1 and I_2 of a commutative ring R with identity 1_R are comaximal if $I_1 + I_2 = R$. If the collection of ideals of R $\{I_i | i = 1,2,\ldots,k\}$ are pairwise comaximal, then $I_1 \cap I_2 \cap \cdots \cap I_k = I_1 I_2 \cdots I_k$ [14]. Now, we modify Corollary 2.4 on [5] in terms of quasi $(m,n)-$closed ideal.

Theorem 3.11. *Let R be a commutative ring with identity 1_R, m and n be natural numbers such that $1 \le n < m$ and $I_1, I_2, \ldots, I_k$ be quasi $(m,n)-$closed ideals, then we have :*

1. *The intersection* $I_1 \cap I_2 \cap \cdots \cap I_k$ *is also a quasi* $(m,n)-$*closed ideal of* R.
2. *If* $I_1, I_2, \ldots, I_k$ *are pairwise comaximal, then* $I_1 I_2 \cdots I_k$ *is a quasi* $(m,n)-$*closed ideal of* R.

Proof. Since Corollary 2.4 on [5] holds for both cases $m \leq 2n$ and $m \geq 2n$, then the first statement is fulfilled. The second statement immediately follows from the first statement. □

Theorem 3.1 can be used to prove the following theorem.

Theorem 3.12. *Let* R *be a commutative ring with identity* 1_R *and* m,n *be natural numbers such that* $1 \leq n < m$. *If* $\{I_\alpha | \alpha \in \Lambda\}$, *where* $\emptyset \neq \Lambda$ *denotes an indexing set, is a collection of quasi* $(m,n)-$*closed ideals of* R, *then* $\cap_{\alpha\in\Lambda} I_\alpha$ *is a quasi* $(m,n)-$*closed ideal of* R.

Proof. Since Theorem 3.1 holds for both cases $m \leq 2n$ and $m \geq 2n$, then it is true that $\cap_{\alpha\in\Lambda} I_\alpha$ is a quasi $(m,n)-$closed ideal of R. □

Let a be an element of a commutative ring R with identity 1_R. If I is an ideal of R, then $I_a = \{x \in R | ax \in I\}$ is an ideal of R. Recall that $a \in R$ is an idempotent element if $a^2 = a$ [12].

Theorem 3.13. *Let* I *be a proper ideal of a commutative ring* R *with identity* 1_R, m *and* n *be natural numbers such that* $1 \leq n < m$ *and* $a \in R - I$ *be a nonunit idempotent element of* R. *If* I *is a quasi* $(m,n)-$*closed ideal of* R, *then* I_a *is a quasi* $(m,n)-$*closed ideal of* R.

Proof. The condition $a \in R - I$ be a nonunit element of R implies that I_a is a proper ideal of R. Let $x \in R$ such that $x^m \in I_a$. Assume that $x^{m-n} \notin I_a$. Consequently, we have $(ax)^m = ax^m \in I$ and $(ax)^{m-n} = ax^{m-n} \notin I$. Since I is a quasi $(m,n)-$closed ideal of R, then $(ax)^n = ax^n \in I$. This means that $x^n \in I_a$. □

The following theorem provide a necessary condition for a proper ideal I of R be a quasi $(m,n)-$closed ideal of R.

Theorem 3.14. *Let* R *be a commutative ring with identity* 1_R *and* m,n *be natural numbers such that* $1 \leq n < m$. *If whenever* $J^m \subseteq I$ *for every ideal* J *of* R *implies* $J^n \subseteq I$ *or* $J^{m-n} \subseteq I$, *then* I *is a quasi* $(m,n)-$*closed ideal of* R.

Proof. Let $x \in R$ such that $x^m \in I$. If $\langle x \rangle$ is an ideal generated by x, then $\langle x \rangle^m \subseteq I$. By assumption, we have $\langle x \rangle^n \subseteq I$ or $\langle x \rangle^{m-n} \subseteq I$. If $\langle x \rangle^n \subseteq I$, then $x^n \in \langle x \rangle^n \subseteq I$. If $\langle x \rangle^{m-n} \subseteq I$, then $x^{m-n} \in \langle x \rangle^{m-n} \subseteq I$. □

Corollary 2.11 on [5] can be modified for quasi $(m,n)-$closed ideal. It appears in Corollary 2 on [6]. Hence, Corollary 3.3 can be extended in terms of quasi $(m,n)-$closed. We omit its proof since it is very similar to Corollary 3.3.

Corollary 3.15. *Let R be a commutative ring with identity 1_R and m, n be natural numbers such that $1 \leq n < m$. If $I[x]$ is a quasi $(m,n)-$closed ideal of $R[x]$, then I is a quasi $(m,n)-$closed ideal of R.*

Corollary 3.4 also holds for quasi $(m,n)-$closed ideal as we can see in Corollary 2 on [6]. Hence, we can extended Theorem 3.5 and Theorem 3.6 in terms of quasi $(m,n)-$closed ideal. Same with above corollary, we omit their proofs.

Theorem 3.16. *Let R be a commutative ring with identity 1_R and m, n be natural numbers such that $1 \leq n < m$. If $I_1, I_2, \ldots, I_p$ are quasi $(m,n)-$closed ideals of R, then $I_1 + I_2 + \cdots + I_p$ is a quasi $(m,n)-$closed ideal of R.*

Theorem 3.17. *Given a proper ideal I of a commutative ring R with identity 1_R and m, n are natural numbers such that $1 \leq n < m$. Ideal $I + \langle x \rangle$ is a quasi $(m,n)-$closed ideal of $R[x]$ if and only if I is a quasi $(m,n)-$closed ideal of R.*

Now, we present Corollary 3 on [6] without proof to support our next result.

Corollary 3.18. *Let I_i be an ideal of commutative ring R_i with identity 1_{R_i} for every $i = 1, 2, \ldots, k$. Let m and n are natural numbers such that $1 \leq n < m$. Then we have :*

1. *If I_i is a proper ideal of R_i for some $i = 1, 2, \ldots, k$, then I_i is a quasi $(m,n)-$closed ideal of R_i if and only if $\prod_{j=1}^{i-1} R_i \times I_i \times \prod_{j=i+1}^{k} R_i$ is a quasi $(m,n)-$closed ideal of $\prod_{i=1}^{k} R_i$.*
2. *If I_i is a quasi $(m_i, n_i)-$closed ideal of R_i for any $i = 1, 2, \ldots, k$ and $t = \max\{n_i, m_i - n_i | i = 1, 2, \ldots, k\}$, then $\prod_{i=1}^{k} I_i$ is a quasi $(m,n)-$closed ideal of $\prod_{i=1}^{k} R_i$ whenever $m \leq \min\{m_1, m_2, \ldots, m_k\}$ and $m \geq 2t$.*

Corollary 3.18 can be used to prove our result.

Corollary 3.19. *Let I_i be an ideal of commutative ring R_i with identity 1_{R_i} for every $i = 1, 2, \ldots, k$ and n be a natural numbers. Then we have :*

1. *If I_i is a proper ideal of R_i for some $i = 1, 2, \ldots, k$, then I_i is a quasi $(2n,n)-$closed ideal of R_i if and only if $\prod_{j=1}^{i-1} R_i \times I_i \times \prod_{j=i+1}^{k} R_i$ is a quasi $(2n,n)-$closed ideal of $\prod_{i=1}^{k} R_i$.*
2. *If I_i is a quasi $(2n,n)-$closed ideal of R_i for any $i = 1, 2, \ldots, k$, then $\prod_{i=1}^{k} I_i$ is a quasi $(2n,n)-$closed ideal of $\prod_{i=1}^{k} R_i$*

Proof. Let $m = 2n$ for every $i = 1, 2, \ldots, k$. Using Corollary 3.18 and facts that $m = 2n = \min\{2n\}$ and $m = 2n = 2\max\{n, 2n - n\} = 2\max\{n\}$, the proof is completely done. □

Example 3.20. *Consider proper ideals of $\mathbb{Z}$, i.e. $\langle 4 \rangle, \langle 9 \rangle$ and $\langle 25 \rangle$. We can check that $\sqrt{\langle 4 \rangle} = \langle 2 \rangle, \sqrt{\langle 9 \rangle} = \langle 3 \rangle$ and $\sqrt{\langle 25 \rangle} = \langle 5 \rangle$. Let $x \in \mathbb{Z}$ such that $x^4 \in \langle 4 \rangle$. Then $x \in \langle 2 \rangle$ and so that $x^2 \in \langle 2 \rangle$. Certainly, this fact also satisfy for $\langle 9 \rangle$ and $\langle 25 \rangle$. Hence, $\langle 4 \rangle, \langle 9 \rangle$ and $\langle 25 \rangle$ are quasi $(2n,n)-$ closed ideals of $\mathbb{Z}$. Using Corollary 3.19, product $\langle 4 \rangle \times \langle 9 \rangle \times \langle 25 \rangle$ is a quasi $(2n,n)-$closed ideal of $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$.*

We already know that ideal $\{0_R\}$ is an $(m, n)-$closed of an integral domain R for every natural numbers m and n such that $1 \leq n < m$. Thus, it is a quasi $(m, n)-$closed ideal of a integral domain R for every natural numbers m and n such that $1 \leq n < m$. Hence, Corollary 3.8 also holds for quasi $(m, n)-$closed ideal. It can be proved with Proposition 3 on [6].

Corollary 3.21. *Let R be a commutative ring with identity 1_R and S be an integral domain. If $f : R \to S$ is a ring homomorphism, then $\ker(f)$ is a quasi $(m, n)-$closed ideal for every natural numbers m and n such that $1 \leq n < m$.*

Recall that a commutative ring R with identity 1_R is called a local ring if R has exactly one maximal ideal. A local ring R with its maximal ideal M is denoted by (R, M) [13]. Hence, we can modify Lemma 2.12 on [7].

Theorem 3.22. *Let m and n be natural numbers such that $1 \leq n < m$. If (R, M) is a local ring which satisfies either $M^n = \{0_R\}$ or $M^{m-n} = \{0_R\}$, then every proper ideal I of R is a quasi $(m, n)-$closed ideal of R.*

Proof. If $M^n = \{0_R\}$ holds, then by Lemma 2.12 on [7], every proper ideal I is an $(m, n)-$closed ideal of R. If $M^{m-n} = \{0_R\}$ holds, then by Lemma 2.12 on [7], every proper ideal I is an $(m, m-n)-$closed ideal of R. Hence, every proper ideal I of R is a quasi $(m, n)-$closed ideal of R. □

4. CONCLUSION

In this paper, we have presented several new properties about $(m, n)-$closed ideal and quasi $(m, n)-$closed ideal. The new properties have been derived from several theorems and corollaries on [5] and [6].

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