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On Graded Strongly 1-Absorbing Primary Ideals and Graded 2-Absorbing I-Primary Ideals

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Abstract. Given a group G with identity e and a $G-$graded commutative ring R with unity element 1_R. This paper introduces a new concept, namely, graded strongly $1-$absorbing primary ideals, which is a subclass of the graded $1-$absorbing primary ideals. A proper graded ideal I of R is said to be a graded strongly $1-$absorbing primary ideal of R if whenever non-unit elements $a, b, c \in h(R)$ with $abc \in I$, then either $ab \in I$ or $c \in Grad(0)$. Several properties of graded strongly $1-$absorbing primary ideals will be investigated in this paper. Furthermore, a new structure called graded $2-$absorbing $I-$primary ideal is also introduced.

Keywords: Graded $1-$Absorbing Primary Ideal, Graded Strongly $1-$Absorbing Primary Ideal, Graded $2-$Absorbing $I-$Primary Ideal.

1. INTRODUCTION

Throughout this article, G will be a group with identity e and R will be an abelian ring with a nonzero unity 1_R. A ring R is said to be $G-$graded if there exists a family of additive subgroups $\{R_g\}_{g\in G}$ such that $R = \bigoplus_{g\in G} R_g$ and $R_{g_1}R_{g_2} \subseteq R_{g_1+g_2}$ for every $g_1, g_2 \in G$. If $R_{g_1}R_{g_2} = R_{g_1+g_2}$ for all $g_1, g_2 \in G$, then R is said to be a strongly graded ring. Moreover, a $G-$graded ring R is positively graded if $R_g = \{0\}$ for all $g < 0$ and negatively graded if $R_g = \{0\}$ for all $g > 0$. The set $h(R) = \bigcup_{g\in G} R_g$ is the set of homogeneous elements of R. A nonzero element $a_g \in R_g$ is said to be a homogeneous element of degree g, and can be written as $\deg(a_g) = g$. Every nonzero element in a $G-$graded ring R can be uniquely expressed as a finite sum of homogeneous elements, denoted by $a = \sum_{g\in G} a_g$, where a_g is a homogeneous component of a in R_g.

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Let S be a subring of a $G-$graded ring R. Then S is said to be a graded subring if $S = \bigoplus_{g\in G} S_g$, where $S_g = S \cap R_g$ for each $g \in G$. Since S is a graded subring of R, then $S_{g_1}S_{g_2} = (S \cap R_{g_1})(S \cap R_{g_2}) \subseteq S \cap R_{g_1+g_2} = S_{g_1+g_2}$ for all $g_1, g_2 \in G$. Consequently, every graded subring S of R is a $G-$graded ring.

Analogous to the concept of an ideal in a ring, the concept of a graded ideal is introduced in graded rings. An ideal I of a graded ring R is said to be a graded ideal if I respects the grading structure, i.e., $I = \bigoplus_{g\in G} I_g$, where $I_g = I \cap R_g$ for each $g \in G$. An important construction related to graded ideals is the graded radical of a graded ideal I, denoted by $Grad(I)$. The graded radical consists of all elements $a = \sum_{g\in G} a_g \in R$ such that for every $g \in G$, there exists a positive integer n_g satisfying $a_g^{n_g} \in I$, denoted by

$$Grad(I) = \left\{ a = \sum_{g\in G} a_g \in R \mid \forall g \in G, \exists n_g \in \mathbb{N}, \ni a_g^{n_g} \in I \right\}.$$

It has been established that the graded radical of a graded ideal is a graded ideal of R.

The following summarizes several important properties related to graded ideals in graded rings.

Lemma 1.1. [1] *Let R be a $G-$graded ring. The following hold:*

(1) If I and J are graded ideals in R, then $I + J$, IJ, and $I \cap J$ are graded ideals in R.
(2) If $a \in h(R)$, then Ra is a graded ideal in R.

Lemma 1.2. [2] *Let R be a $G-$graded ring, and let I, J be graded ideals in R. Then,*

(1) $Grad(Grad(I)) = Grad(I)$
(2) $Grad(IJ) = Grad(I \cap J) = Grad(I) \cap Grad(J)$.

Proof. It follows from ([2], Proposition 2.4). □

Refai, in [3], introduced a generalization of the graded prime ideal, called the graded primary ideal. A graded ideal I of a $G-$graded ring R is said to be a graded primary ideal if $I \neq R$ and whenever $a, b \in h(R)$ with $ab \in I$, then $a \in I$ or $a \in Grad(I)$. Every graded prime ideal is a graded primary ideal, but the converse does not generally hold. Later, in [4], a further generalization of the graded primary ideal, called the graded $2-$absorbing primary ideal, was studied. A proper graded ideal I of R is said to be a graded $2-$absorbing primary ideal if whenever $a, b, c \in h(R)$ with $abc \in I$, then $ab \in I$ or $ac \in Grad(I)$ or $bc \in Grad(I)$. In [5], Abu-Dawwas and Bataineh introduced a new subclass of graded $2-$absorbing primary ideals, called graded $1-$absorbing primary ideals. A proper graded ideal I of a graded ring R is said to be a graded $1-$absorbing primary if whenever non-unit elements $a, b, c \in h(R)$ such that $abc \in I$, then $ab \in I$ or $c \in Grad(I)$. Since the concept of a graded $1-$absorbing primary ideal generalizes that of a graded primary ideal, it follows that every graded primary ideal is necessarily a graded

$1-$absorbing primary ideal. However, the converse is not necessarily true; that is, a graded $1-$absorbing primary ideal is not always a graded primary ideal.

In 2015, [6] studied rings in which every non-unit element is a product of a unit and a nilpotent element, referring to them as UN rings. Building upon this concept, the structure of the UN rings was later extended to graded rings.

Definition 1.3. [7] A $G-$graded ring R is called a HUN ring if every homogeneous element in R is either a unit or nilpotent.

Example 1.4. [8] Let R be a graded field, and let $u \notin R$ be an element such that $u^2 = 1$. Define a graded field $F = \{a + ub \mid a, b \in R \text{ and } u^2 = 1\}$ with respect to the group $\mathbb{Z}_2$, where the grading is given by $F_0 = R$ and $F_1 = uR$. Next, we prove that F is a HUN-ring. Let $a \in F_0$. Since every element of the field R is a unit, it follows that a is a unit in F_0. Now, consider the elements in F_1. Take any $ub \in F_1$ with $u^2 = 1$. Since $b \in R$ and b is a unit, we obtain $(ub)^2 = u^2b^2 = b^2$. Consequently, b^2 is trivially nilpotent if $b = 0$. Conversely, if $b \neq 0$, then b^2 is a unit. Therefore, the graded field F is a HUN ring.

Previously, in [9], Almahdi et al. introduced the concept of a strongly $1-$absorbing primary ideal, which can be used to characterize UN rings and local rings, rings that have exactly one maximal ideal. Based on this idea, in [8], Abu-Dawwas developed a similar concept in graded rings, introducing the graded strongly $1-$absorbing primary ideal, which forms a new subclass of graded $1-$absorbing primary ideals. Just like the strongly $1-$absorbing primary ideal, the graded strongly $1-$absorbing primary ideal can also be used to characterize HUN rings and graded local rings.

This paper is a review of the work previously written by Abu-Dawwas in [8]. Our contributions in this paper include providing examples, adding properties, and deriving corollaries from the existing propositions. We give examples (see Example 2.2) of a graded strongly $1-$absorbing primary ideal, and derive corollaries from Proposition 2.12, particularly when R is a graded Noetherian. Furthermore, we investigate the structure of graded ideals in the direct product of graded rings. Since the graded ideal in this ring is not a graded strongly $1-$absorbing primary ideal, we examine whether it satisfies the properties of a graded $2-$absorbing $I-$ideal, which is a generalization of graded prime ideals and was introduced by I. Akray, Adil K. Jabbar and Shadan A. Othman in [10].

2. GRADED STRONGLY $1-$ABSORBING PRIMARY IDEAL

In this section, we study the concept of graded strongly $1-$absorbing primary ideals, a subclass of graded $1-$absorbing primary ideals.

Definition 2.1. [8] *Let R be a $G-$graded ring and I a proper graded ideal of R. The ideal I is said to be a graded strongly $1-$absorbing primary ideal in R if whenever non-unit elements $a, b, c \in h(R)$ such that $abc \in I$, then either $ab \in I$ or $c \in Grad(\{0\})$.*

The following example illustrates the concept of a graded strongly $1-$absorbing primary ideal.

Example 2.2. *Let $R = \mathbb{R}[x]/\langle x^9 \rangle$ be a $\mathbb{Z}-$graded ring and consider the graded ideal $I = \langle \overline{x^3} \rangle$ in R. Then I is a graded strongly $1-$absorbing primary ideal. Let $\overline{p(x)} = \overline{\sum_{i=0}^{8} a_i x^i}, \overline{q(x)} = \overline{\sum_{i=0}^{8} b_i x^i}, \overline{r(x)} = \overline{\sum_{i=0}^{8} c_i x^i} \in h(R)$ be non-unit elements such that $\overline{p(x)q(x)r(x)} \in I$. If $\overline{p(x)q(x)} \in I$, then I is trivially a graded strongly $1-$absorbing primary ideal. Suppose $\overline{p(x)q(x)} \notin I$. Then we analyze the following cases:*

1. *If $\deg(\overline{p(x)q(x)}) = 0$, then $\deg(\overline{r(x)}) \geq 3$, so there exists $n_i \geq 3$ such that $(\overline{r(x)})^{n_i} \in \{\overline{0}\}$. Hence, $\overline{r(x)} \in \mathrm{Grad}(\{\overline{0}\})$.*
2. *If $\deg(\overline{p(x)q(x)}) = 1$, then $\deg(\overline{r(x)}) \geq 2$, so there exists $n_i \geq 5$ such that $(\overline{r(x)})^{n_i} \in \{\overline{0}\}$. Hence, $\overline{r(x)} \in \mathrm{Grad}(\{\overline{0}\})$.*
3. *If $\deg(\overline{p(x)q(x)}) = 2$, then $\deg(\overline{r(x)}) \geq 1$, so there exists $n_i \geq 9$ such that $(\overline{r(x)})^{n_i} \in \{\overline{0}\}$. Hence, $\overline{r(x)} \in \mathrm{Grad}(\{\overline{0}\})$.*

Thus, for any $\overline{p(x)}, \overline{q(x)}, \overline{r(x)} \in h(R)$ such that $\overline{p(x)q(x)r(x)} \in I$, we have either $\overline{p(x)q(x)} \in I$ or $\overline{r(x)} \in \mathrm{Grad}(\{\overline{0}\})$. Therefore, I is a graded strongly $1-$absorbing primary ideal in R. In general, the ideal $I = \langle \overline{x^p} \rangle$ is a graded strongly $1-$absorbing primary ideal of $R = \mathbb{R}[x]/\langle x^{p^n} \rangle$.

As mentioned earlier, the concept of a graded strongly $1-$absorbing primary ideal forms a new subclass of graded $1-$absorbing primary ideals. Hence, every graded strongly $1-$absorbing primary ideal is a graded $1-$absorbing primary ideal, but the converse does not necessarily hold. Not every graded $1-$absorbing primary ideal is graded strongly $1-$absorbing. The following example illustrates this.

Example 2.3. *Let $R = \mathbb{Z}[i]$ and $G = \mathbb{Z}_2$. Then R be a $G-$graded ring by $R_0 = \mathbb{Z}$ and $R_1 = i\mathbb{Z}$. Consider the graded ideal $I = 3R$, which is a graded $1-$absorbing primary ideal. For arbitrary non-unit elements $a, b \in h(R)$ with $ab \in I$ and $a \notin I$, it follows that 3 divides ab but does not divide a. Thus, 3 must divide b, which means $b \in Grad(I)$. Hence, I is a graded $1-$absorbing primary ideal. However, I is not a graded strongly $1-$absorbing primary ideal. Specifically, consider $2, 3 \in h(R)$ such that $2 \cdot 2 \cdot 3 \in I$, but $2 \cdot 2 \notin I$ and $3 \notin Grad(\{0\})$. Therefore, we conclude that I is a graded $1-$absorbing primary ideal but not a graded strongly $1-$absorbing primary ideal.*

Not every $G-$graded ring necessarily contains a graded strongly $1-$absorbing primary ideal. However, a $G-$graded ring does contain a graded strongly $1-$absorbing primary ideal if it has the graded prime ideal $Grad(\{0\})$ or is a graded local ring. This result is formalized in the following theorem.

Theorem 2.4. [8] *Let R be a $G-$graded ring. A graded strongly $1-$absorbing primary ideal exists in R if and only if*

1. *the ideal $Grad(0)$ is a graded prime ideal, or*
2. *the ring R is a graded local ring.*

Proof. It follows from ([8], Theorem 2.8.). □

In ring theory, if R and S are rings, then the direct product $R \times S$ forms a ring. Similarly, if R and S are $G-$graded rings with grading $\{R_g\}_{g\in G}$ and $\{S_g\}_{g\in G}$, then the direct product $R \times S$ is also a $G-$graded ring with grading $(R \times S)_g = R_g \times S_g$ for each $g \in G$.

Corollary 2.5. [8] *If R and S are $G-$graded rings, then $R \times S$ does not contain any graded strongly $1-$absorbing primary ideal.*

Proof. Let R and S be $G-$graded rings. If neither R nor S is a graded local ring, then it is clear that $R \times S$ is also not a graded local ring. Suppose that R and S are graded local rings with graded maximal ideals $\mathcal{M}_R$ and $\mathcal{M}_S$, respectively. Consequently, $R \times S$ has more than one graded maximal ideal, namely $\mathcal{M}_R \times S$ and $R \times \mathcal{M}_S$. Therefore, $R \times S$ is not a graded local ring. Next, we examine whether $Grad(\{0_{R\times S}\}) = Grad(\{0_R\}) \times Grad(\{0_S\})$ is a graded prime ideal in $R \times S$. Consider elements $(a, b) \notin Grad(\{0_{R\times S}\})$ where $a \in Grad(\{0_R\})$ and $b \notin Grad(\{0_S\})$, as well as $(c, d) \notin Grad(\{0_{R\times S}\})$ where $c \notin Grad(\{0_R\})$ and $d \in Grad(\{0_S\})$. However, since there exist $m, n \in \mathbb{N}$ such that $(ac)^n = a^n c^n = 0_R c^n = 0_R$ and $(bd)^m = b^m d^m = b^m 0_S = 0_S$, it follows that $(ac, bd) \in Grad(\{0_{R\times S}\})$. As a result, $Grad(\{0_{R\times S}\})$ is not a graded prime ideal in $R \times S$. Since $R \times S$ is not a graded local ring and $Grad(\{0_{R\times S}\})$ is not a graded prime ideal, we conclude that $R \times S$ does not contain any graded strongly $1-$absorbing primary ideal. □

Some properties of graded strongly $1-$absorbing primary ideals are presented in the following propositions.

Proposition 2.6. [8] *Let R be a $G-$graded ring, and let I and J be proper graded ideals of R. If I and J are graded strongly $1-$absorbing primary ideals, then $I \cap J$ is also a graded strongly $1-$absorbing primary ideal.*

Proof. It follows from ([8], Proposition 2.12). □

The following theorem is a development of the results studied in [11],[3], which were previously investigated in the context of $1-$absorbing primary ideals and graded primary ideals. In this study, we further develop these results within the framework of graded strongly $1-$absorbing primary ideals.

Definition 2.7. *Let R be a $G-$graded ring and let I be a graded strongly $1-$absorbing primary ideal of R. Then $J = Grad(I)$ is a graded prime ideal of R, and we say that I is a $J-$graded strongly $1-$absorbing primary ideal.*

So we have the following result.

Proposition 2.8. *Let R be a $G-$graded ring, and let $I_1, I_2, \ldots, I_n$ be proper graded ideal of R. If $I_1, I_2, \ldots, I_n$ are $J-$graded strongly $1-$absorbing primary ideal, then $I = \cap_{i=1}^n I_i$ is a $J-$graded strongly $1-$absorbing primary ideal.*

Proof. First, we will prove that $Grad(I) = J$. Let $I_1, I_2, \ldots, I_n$ are J-graded strongly 1-absorbing primary ideal. It is given that $Grad(I_i) = J$ for every $i = 1, 2, \ldots, n$. By Proposition 1.2, we have $Grad(I) = Grad\left(\bigcap_{i=1}^{n} I_i\right) = \bigcap_{i=1}^{n} Grad(I_i)$ Since $Grad(I_i) = J$ for all i, it follows that $\bigcap_{i=1}^{n} Grad(I_i) = J \cap J \cap \cdots \cap J = J$. Thus, we have shown that $Grad(I) = J$. Next, by Proposition 2.6, we know that $I = \bigcap_{i=1}^{n} I_i$ is a graded strongly $1-$absorbing primary ideal of R. Therefore, we conclude that I is a $J-$graded strongly $1-$absorbing primary ideal. □

Proposition 2.9. [8] *Let R be a $G-$graded ring. If every element of $h(R)$ is either nilpotent or a unit, then Rw is a graded strongly $1-$absorbing primary ideal in R for every non-unit element $w \in h(R)$.*

Proof. It follows from ([8], Proposition 2.13). □

Corollary 2.10. [8] *Let R be a graded ring. If R is an HUN ring, then every proper graded ideal in R is a graded strongly $1-$absorbing primary ideal.*

Proof. Let R be a HUN ring and I a proper graded ideal in R. By Definition 1.3, every element in $h(R)$ is either nilpotent or a unit. Suppose there exist nonunit elements $a, b, c \in h(R)$ such that $abc \in I$ and $c \notin Grad(\{0\})$. The objective is to show that every proper graded ideal I in R is a graded strongly $1-$absorbing primary ideal. Since $a, b, c \in h(R)$ are nonunit elements and $abc \in R(abc)$, Proposition 2.9 ensures that $R(abc)$ is a graded strongly $1-$absorbing primary ideal in R. Consequently, $ab \in R(abc) \subseteq I$. Therefore, it is established that if every element in $h(R)$ is either nilpotent or a unit, then every proper graded ideal in R is a graded strongly $1-$absorbing primary ideal. □

Example 2.11. *Consider the ring $\mathbb{Z}/9\mathbb{Z}$. The following table demonstrates that every element in $\mathbb{Z}/9\mathbb{Z}$ is either a unit or nilpotent.*

TABLE 1. Multiplication $(\cdot)$ in $\mathbb{Z}/9\mathbb{Z}$.

$\cdot$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$	$\overline{6}$	$\overline{7}$	$\overline{8}$
$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$
$\overline{1}$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$	$\overline{6}$	$\overline{7}$	$\overline{8}$
$\overline{2}$	$\overline{0}$	$\overline{2}$	$\overline{4}$	$\overline{6}$	$\overline{8}$	$\overline{1}$	$\overline{3}$	$\overline{5}$	$\overline{7}$
$\overline{3}$	$\overline{0}$	$\overline{3}$	$\overline{6}$	$\overline{0}$	$\overline{3}$	$\overline{6}$	$\overline{0}$	$\overline{3}$	$\overline{6}$
$\overline{4}$	$\overline{0}$	$\overline{4}$	$\overline{8}$	$\overline{3}$	$\overline{7}$	$\overline{2}$	$\overline{6}$	$\overline{1}$	$\overline{5}$
$\overline{5}$	$\overline{0}$	$\overline{5}$	$\overline{1}$	$\overline{6}$	$\overline{2}$	$\overline{7}$	$\overline{3}$	$\overline{8}$	$\overline{4}$
$\overline{6}$	$\overline{0}$	$\overline{6}$	$\overline{3}$	$\overline{0}$	$\overline{6}$	$\overline{3}$	$\overline{0}$	$\overline{6}$	$\overline{3}$
$\overline{7}$	$\overline{0}$	$\overline{7}$	$\overline{5}$	$\overline{3}$	$\overline{1}$	$\overline{8}$	$\overline{6}$	$\overline{4}$	$\overline{2}$
$\overline{8}$	$\overline{0}$	$\overline{8}$	$\overline{7}$	$\overline{6}$	$\overline{5}$	$\overline{4}$	$\overline{3}$	$\overline{2}$	$\overline{1}$

Since every ring can be viewed as a trivially graded ring, $\mathbb{Z}/9\mathbb{Z}$ can be considered as a $\mathbb{Z}-$graded ring with grading $R_0 = \mathbb{Z}/9\mathbb{Z}$ and $R_n = \{\overline{0}\}$ for every $n \in \mathbb{Z}, n \neq 0$. Thus, every homogeneous element of $\mathbb{Z}/9\mathbb{Z}$ is either a unit or nilpotent. By

Corollary 2.10, every proper graded ideal in $\mathbb{Z}/9\mathbb{Z}$ is a graded strongly $1-$absorbing primary ideal.

The following provides necessary and sufficient conditions for a graded prime ideal of a graded ring to be a graded strongly $1-$absorbing primary ideal.

Proposition 2.12. [8] *Let R be a $G-$graded ring. A graded prime ideal in R is a graded strongly $1-$absorbing primary ideal if and only if*

(1) R is a HUN-ring, or
(2) R is a graded local ring with a graded maximal ideal M, has exactly one graded prime ideal that is not the graded maximal ideal (i.e., $Grad(\{0\})$), and every graded M-primary ideal contains M^2.

Proof. It follows from ([8], Proposition 2.17). □

Next, we extend the results of [9], originally on strongly $1-$absorbing primary ideals, to the graded case. A $G-$graded ring R is called a graded Noetherian if every ascending chain of graded ideals in R terminates. Thus, we obtain the following consequence of Proposition 2.12.

Corollary 2.13. *For any graded Noetherian ring R, the following are equivalent :*

(1) Every graded primary ideal of R is graded strongly $1-$absorbing primary.
(2) R is a HUN ring.

Proof. (1) ⇒ (2) Suppose that R is not a HUN ring. Then, by Proposition 2.12, R is a graded local ring with maximal ideal M, and every graded M-primary ideal contains M^2. By Proposition 1.2, we obtain

$$\begin{aligned} Grad(M^4) &= Grad(M) \cap Grad(M) \cap Grad(M) \cap Grad(M) \\ &= Grad(M) \\ &= Grad(Grad(I)), \text{for some graded } M-\text{primary ideal } I \\ &= Grad(I) \\ &= M. \end{aligned}$$

Thus, M^4 is an $M-$primary ideal, and since $M^2 \subseteq M^4$, we get $M^2 = M^4$. Consequently, M^2 is an idempotent ideal. Since R is Noetherian, M^2 is generated by an idempotent element of R. However, because R is a graded local ring, the only idempotent elements in R are 0 and 1. Therefore, we conclude that $M^2 = \{0\}$. As a result, $M^2 \subseteq \text{Grad}(\{0\})$, which implies $M = \text{Grad}(\{0\})$. This means that R is a HUN ring, contradicting our initial assumption. Hence, we conclude that R must be a HUN ring.
(2) ⇒ (1) If R is a graded Noetherian ring and a HUN ring, then by Corollary 2.10 every ideal of R is graded strongly $1-$absorbing primary. Similarly, every graded primary ideal is also graded strongly $1-$absorbing primary. □

Let R be a commutative ring with identity 1_R and let M be an R-module. The idealization of the module M (trivial extension of the ring R by the module M), denote by

$$R(+)M = \{(r, m) \mid r \in R, m \in M\},$$

is a commutative ring with identity $(1_R, 0)$, equipped with componentwise addition and multiplication $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2+r_2m_1)$ for all $(r_1, m_1), (r_2, m_2) \in R(+)M$. If $R = \bigoplus_{g\in G} R_g$ be a $G-$graded ring and $M = \bigoplus_{g\in G} M_g$ be a $G-$graded R-module, then $R(+)M$ is a $G-$graded ring with grading $\{R_g(+)M_g\}_{g\in G}$. Suppose that I is an ideal of R and N is a submodule of M. Then $I(+)N$ is an ideal of $R(+)M$ if and only if $IM \subseteq N$. In the context of graded rings, $I(+)N$ is a graded ideal of $R(+)M$ if and only if I is a graded ideal of R and N is a graded submodule of M. Furthermore, the graded radical of a graded $I(+)N$ is defined by $Grad(I(+)N) = Grad(I)(+)M$. The following proposition provides a necessary condition for a homogeneous graded ideal $I(+)N$ to be a graded strongly $1-$absorbing primary ideal in $R(+)M$.

Proposition 2.14. *Let M be a graded R-module, and let $I(+)N$ be a homogeneous graded ideal of $R(+)M$. If $I(+)N$ is a graded strongly $1-$absorbing primary ideal in $R(+)M$, then I is a graded strongly $1-$absorbing primary ideal of R.*

Proof. Let $i_1, i_2, i_3 \in h(R)$ be homogeneous non-unit elements such that $i_1i_2i_3 \in I$ and $i_3 \notin Grad(\{0_R\})$. Then $(i_1, 0)(i_2, 0)(i_3, 0) = (i_1i_2i_3, 0) \in I(+)N$. Since $I(+)N$ is a graded strongly $1-$absorbing primary ideal and $(1_3, 0) \notin Grad(\{0_{R(+)M}\})$, it follows that $(i_1, 0)(i_2, 0) = (i_1i_2, 0) \in I(+)N$, which implies $i_1i_2 \in I$. Hence, I is a graded strongly $1-$absorbing primary ideal of R. $\square$

In the following, we introduce a class of ideals that generalizes the concept of graded prime ideals, namely graded $2-$absorbing $I-$ideal, where I is a fixed proper ideal.

Definition 2.15. [10] *Let R be a $G-$graded ring and let I be a fixed proper ideal of R_e. A proper graded ideal P of R is called a graded $2-$absorbing $I-$ideal ideal if for all $a, b, c \in h(R)$ such that $abc \in P - IP$, then $ab \in P$ or $ac \in P$ or $bc \in P$.*

Theorem 2.16. [10] *If P and Q are non zero graded $I-$prime ideals of a $G-$graded ring R, then $P \cap Q$ is a graded $2-$absorbing $I-$ideal.*

Proof. It follows from ([10], Theorem 2.4). $\square$

Proposition 2.17. *Let P be graded strongly $1-$absorbing primary ideal of a $G-$graded ring R and I be a graded ideal. Then $Grad(P)$ is a graded $I-$prime ideal of R.*

Proof. Let $a, b \in h(R)$ be arbitrary non-unit homogeneous elements such that $ab \in Grad(P) - IGrad(P)$, which means that $ab \in Grad(P)$ and $ab \notin IGrad(P)$. Therefore, there exists $n \in \mathbb{N}$ such that $(ab)^n = a^nb^n \in P$. Suppose $n = r + s$ for some $r, s \in \mathbb{N}$. Since P is a graded strongly $1-$absorbing primary ideal, it follows that $a^ra^s = a^n \in P$ or $b^n \in Grad(\{0\}) \subseteq Grad(P)$. In other words, either $a^n \in P$

or $b^{nm} \in P$ for some $m \in \mathbb{N}$. Consequently, $a \in Grad(P)$ or $b \in Grad(P)$. Hence, it is concluded that $Grad(P)$ is a graded $I-$prime ideal. $\square$

Proposition 2.18. *Let P and Q be a graded strongly $1-$absorbing primary ideal of R. Then $Grad(PQ)$ is a graded $2-$absorbing $I-$ideal of R.*

Proof. Based on Lemma 1.2, it follows that

$$Grad(PQ) = Grad(P \cap Q) = Grad(P) \cap Grad(Q).$$

Since P and Q are graded strongly $1-$absorbing primary ideals, then by Proposition 2.17, both $Grad(P)$ and $Grad(Q)$ are graded $I-$prime ideals of R. Furthermore, by Theorem 2.16, $Grad(PQ)$ is a graded $2-$absorbing $I-$ideal. $\square$

It has been shown that the $G-$graded ring $R \times S$ does not admit any graded strongly $1-$absorbing primary ideal. Based on Definition 2.15, we investigate whether the direct product $P \times Q$ forms a graded $2-$absorbing $(I \times J)-$ideal in $R \times S$, where P and Q are graded strongly $1-$absorbing primary ideals in R and S, respectively. Let $(a_1, a_2), (b_1, b_2), (c_1, c_2) \in h(R \times S)$ such that

$$(a_1, a_2)(b_1, b_2)(c_1, c_2) = (a_1b_1c_1, a_2b_2c_2) \in (P \times Q) - (I \times J)(P \times Q),$$

which means that $(a_1b_1c_1, a_2b_2c_2) \in P \times Q$ and $(a_1b_1c_1, a_2b_2c_2) \notin (I \times J)(P \times Q)$. Since P and Q are graded strongly $1-$absorbing primary ideals in R and S, respectively, it follows that $a_1b_1 \in P$ or $c_1 \in Grad(\{0_R\})$, and $a_2b_2 \in Q$ or $c_2 \in Grad(\{0_S\})$. In other words, $(a_1b_1, a_2b_2) \in P \times Q$ or $(c_1, c_2) \in Grad(\{0_R\}) \times Grad(\{0_S\})$. Since $Grad(\{0_R\}) \subseteq Grad(P)$ and $Grad(\{0_S\}) \subseteq Grad(Q)$, we obtain $(a_1b_1, a_2b_2) \in P \times Q$ or $(a_1c_1, a_2c_2) \in Grad(P) \times Grad(Q)$ or $(b_1c_1, b_2c_2) \in Grad(P) \times Grad(Q)$. This result shows that in general $P \times Q$ does not satisfy the definition of a graded $2-$absorbing $(I \times J)-$ideal. This motivates introducing a new class of graded ideals called graded $2-$absorbing $I-$primary ideals.

Definition 2.19. *Let R be a $G-$graded ring and I a fixed proper ideal of R_e. A proper graded ideal P of R is called a graded $2-$absorbing $I-$primary ideal if for all $a, b, c \in h(R)$ with $abc \in P - IP$, then $ab \in P$ or $ac \in Grad(P)$ or $bc \in Grad(P)$.*

3. CONCLUSIONS

This article establishes several key results. The intersection of any collection of $J-$graded strongly $1-$absorbing primary ideals is again a $J-$graded $1-$absorbing primary ideal. A graded Noetherian ring is a $HUN-$ring if and only if every graded primary ideal is a graded strongly $1-$absorbing primary ideal. A homogeneous graded ideal of the form $I(+)N$ is a graded strongly $1-$absorbing primary ideal in the graded idealization $R(+)M$ if and only if I is a graded strongly $1-$absorbing primary ideal in R. Furthermore, we introduced generalizations of graded prime ideals, namely graded $I-$prime ideals and graded $2-$absorbing $I-$ideal, where I is a fixed proper ideal of R. For any graded strongly $1-$absorbing primary ideal in a graded ring R, its graded radical is a graded $I-$prime ideal, and the graded radical of the product of two graded strongly $1-$absorbing primary ideals is a graded $2-$absorbing $I-$ideal. Finally, even if graded rings R and S each contain graded strongly $1-$absorbing primary ideals, their direct product $R \times S$ does not necessarily admit such an ideal. However, it was shown that a graded ideal of the form $P \times Q$ in $R \times S$, where P and Q are graded strongly $1-$absorbing primary ideals in R and S, respectively, forms a graded $2-$absorbing $I-$primary ideal.

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