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ON LEFT QUASI-EXACT SEQUENCES

(SEPUTAR BARISAN QUASI EKSAK KIRI)

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Abstract. The aim of this paper is to bring together the properties of quasi-exact sequences and homomorphism functors. It is of interest to know whether the left quasi-exact sequences have some interesting properties related to the functors and vice versa. Moreover, we also define the dualization of left quasi-exact sequences, i.e. right quasi-coexact sequences. Our proofs show that the quasi-exactness of sequences is preserved by any homomorphism functors.

Keywords: Exact sequences, quasi-exact sequences, left quasi-exact sequences, right quasi-coexact sequences, homomorphism functors.

Abstrak. Salah satu sifat barisan modul-modul dan homomorfisma adalah keeksakan barisan tersebut terkait dengan sifat fungtor homomorfisma yang dikenakan padanya. Barisan quasi eksak kiri merupakan perumuman barisan eksak kiri. Dalam penelitian ini dilihat hubungan barisan quasi eksak kiri dengan fungtor homomorfisma. Selanjutnya didefinisikan pula dualisasinya yaitu barisan quasi koeksak kanan. Penelitian ini membuktikan bahwa sifat barisan quasi eksak kiri maupun kanan dipertahankan oleh fungtor homomorfisma.

Kata-kata kunci: Barisan eksak, barisan quasi eksak, barisan quasi eksak kiri, barisan quasi eksak kanan, fungtor homomorfisma.

1 INTRODUCTION

Exact sequences play important roles in module theory and homology. Some notions such as injectivity and projectivity of modules, generators and subgenerators of modules, have been defined and analyzed by exact sequence approach (see [6]). Generalizing exact sequences to quasi-exact sequences gives possibilities to generalized some related notions which are defined by exact sequences approach. In this work, we continue to observe further properties of quasi-exact sequences introduced by Anvariyeh and Davvaz ([2] and [3]). We will restrict our discussion to left quasi-exact sequences as a generalization of left exact sequences. It is our

2010 Mathematics Subject Classification: 16D10 purpose to study the applications of homomorphism functors to left quasi-exact sequences. It is of interest to know whether the left quasi-exact sequences have some interesting properties related to the functors and vice versa. We also define the dualization of left quasi-exact sequences, i.e. right quasi-coexact sequences.

Let R be an associative ring with unit and the modules mean left unital R-modules. We consider some well known exactness of sequences. A more complete explanation may be obtained by Adkins and Weintarub [1] or Wisbauer [6]. A sequence of R-modules and R-homomorphisms

$$\ldots \xrightarrow{f_{i-1}} A_{i-1} \xrightarrow{f_i} A_i \xrightarrow{f_{i+1}} A_{i+1} \xrightarrow{f_{i+2}} \ldots$$

is called an exact sequence in A_i if $Im(f_i) = Ker(f_{i+1})$. This sequence is called an exact sequence if it is exact in A_i for every i. For a special case, if the sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$

is exact, then f is injective, $Im(f) = Ker(g)$, $Im(g) = Ker(h)$, and we call it a left exact sequence. On the other hand, if the sequence

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \longrightarrow 0$$

is exact, then $Im(f) = Ker(g)$, $Im(g) = Ker(h)$, h is surjective, and we call it a right exact sequence.

Anvariyeh and Davvaz in [5] introduced a quasi-exact sequence which is a generalization of exact sequences. A sequence of R-modules and R-homomorphisms $A \xrightarrow{f} B \xrightarrow{g} C$ is quasi-exact in B or U-exact in B if there exists a submodule U in C such that $Im(f) = g^{-1}(U)$. A sequence $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is called a short U-exact sequence or short quasi-exact sequence if f is injective, g is surjective and $Im(f) = g^{-1}(U)$ for some submodule U in C [3]. Anvariyeh and Davvaz studied some properties of short U-exact sequence such as the generalization of Five Lemma, ascending and descending chain of modules in sequences [5]. Moreover, Anvariyeh and Davvaz observed the dual of quasi-exact sequences and U-split sequences [2]. The generalization of Snake Lemma was observed by Davvaz and Solt in quasi-exact sequences and they gave some results in concepts of generalization in algebra homology [4]. Furthermore, Anvariyeh and Davvaz investigated some properties of finitely generated modules, essential submodules, small submodules and Schanuel Lemma in a short U-exact sequence [3].

Consider now the following sequence:

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \qquad (1.1)$$

which is U_B-exact in A, U_C-exact in B, and U_D-exact di C, for some $U_B \subseteq B$, $U_C \subseteq C$ and $U_D \subseteq D$. Based on the definition of quasi-exact sequence, it yields $0 =$

$f^{-1}(U_B)$. It is clear that $Ker(f) \subseteq f^{-1}(U_B) = 0$, so $Ker(f) = 0$ and moreover f is injective. We call such sequence as a left quasi-exact sequence. If in the Sequence (1.1) we put $D = 0$, then $U_D = 0$, so g is surjective and the Sequence (1.1) is a short U_C-exact sequence. Hence the short quasi-exact sequence is a special case of a left quasi-exact sequence.

Now we recall the definition of V-coexact sequences as a dualization of quasi-exact sequences.

Definition 1.1. [3] *A sequence of R-modules $A \xrightarrow{f} B \xrightarrow{g} C$ is called V-coexact in B if there exists a submodule V of A such that $f(V) = Ker(g)$.*

Any U-exact sequence $A \xrightarrow{f} B \xrightarrow{g} C$ is a $Ker(gf)$-coexact sequence. Conversely, any V-coexact sequence $A \xrightarrow{f} B \xrightarrow{g} C$ is an $Im(gf)$-exact sequence. Moreover, a sequence

$$\dots \xrightarrow{f_{i-1}} A_{i-1} \xrightarrow{f_i} A_i \xrightarrow{f_{i+1}} A_{i+1} \xrightarrow{f_{i+2}} \dots$$

is called dual quasi-exact if it is V_{i-1}-coexact in A_i for every i, where V_{i-1} is a submodule of A_{i-1}.

As a dual of quasi-exact sequence, every short U-exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is a short $Ker(gf)$-coexact sequence. Conversely, every short V-coexact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is a short $Im(gf)$-exact sequence. Hence, we can apply some properties of a short U-exact sequence for a short V-coexact sequence.

In the next section we discuss the notions of left quasi-exact sequences and the implications of applying hom-functor to the sequence. Our main results will be found in Proposition 2.5 and Proposition 2.6. The last section presents an introduction of right quasi-coexact sequences as the dual of left quasi-exact sequences.

2 LEFT QUASI-EXACT SEQUENCES

A sequence $0 \longrightarrow A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \xrightarrow{f_3} A_4 \xrightarrow{f_4} \dots$ is quasi-exact if f_1 injective and it is U_{i+1}-exact in A_i for every $i \geqslant 2$, where U_{i+1} is a submodule of A_{i+1}. Now we give the definition of a left quasi-exact sequence.

Definition 2.1. *A sequence $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ is called a left quasi-exact sequence if f is injective and it is U_C-exact in B and U_D-exact in C for some U_C submodule of C and U_D submodule of D.*

Example 2.2. *The sequence of* $\mathbb{Z}$*-modules* $0 \longrightarrow \mathbb{Z}_2 \xrightarrow{i_1} \mathbb{Z}_4 \xrightarrow{i_2} \mathbb{Z}_8 \xrightarrow{i_3} \mathbb{Z}_{16}$ *is a left quasi-exact sequence, where* $i_1(\bar{a}) = 2\bar{a}$ *for all* $\bar{a} \in \mathbb{Z}_2$*,* $i_2(\bar{b}) = 2\bar{b}$ *for all* $\bar{b} \in \mathbb{Z}_4$*, and* $i_3(\bar{c}) = 2\bar{c}$ *for all* $\bar{c} \in \mathbb{Z}_8$*.*

Moreover, a sequence $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ is left quasi-exact if f is injective, $Im(f) = g^{-1}(U_C)$, and $Im(g) = h^{-1}(U_D)$ for some submodule U_C in C and U_D in D. We get a special case when $D = 0$ as we consider in the following equivalence conditions.

Lemma 2.3. *A sequence*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \tag{2.1}$$

is a left quasi-exact sequence which is D*-exact in* C *if and only if the sequence*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} 0 \tag{2.2}$$

is short U_C*-exact for a submodule* U_C *in* C*.*

Proof. ($\Rightarrow$) Based on the assumption f is injective, $Im(f) = g^{-1}(U_C)$ for some U_C in C and $Im(g) = h^{-1}(D)$. It is clear that $h^{-1}(D) = C$, so $Im(g) = C$ and g is surjective. Then we form the Sequence (2.2) where f is injective, g is surjective, and $Im(f) = g^{-1}(U_C)$ for some submodule U_C di C. Hence the sequence is short U_C-exact.

($\Leftarrow$) Based on the assumption f is injective, g is surjective and $Im(f) = g^{-1}(U_C)$ for a submodule U_C in C, so $Im(g) = C$. For any R-homomorphism $h : C \longrightarrow D$, $h^{-1}(D) = C = Im(g)$. It implies that the Sequence (2.1) is left quasi-exact, which is D-exact in C. $\square$

Consider now for any submodule L of B, a left quasi-exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$

induces another left quasi-exact sequence, as we give in the following proposition.

Proposition 2.4. *Let* $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ *be a left quasi-exact sequence,* L *be a submodule of* B*. If* $M = g(L)$*,* $K = f^{-1}(f(A) \cap L)$*,* $N = hg(L)$*,* $f_1 = f\mid_K$*,* $g_1 = g\mid_L$*, and* $h_1 = h\mid_M$*, then the sequence*

$$0 \longrightarrow K \xrightarrow{f_1} L \xrightarrow{g_1} M \xrightarrow{h_1} N$$

is left quasi-exact.

Proof. Based on Definition 2.1, f is injective and $f_1 = f\mid_K$ is injective. Moreover, $Im(f) = g^{-1}(U_C)$ and $Im(g) = h^{-1}(U_D)$ for submodules U_C in C and U_D in D.

Now take $X = g(g^{-1}(U_C) \cap L)$ and we want to show that $Im(f_1) = {g_1}^{-1}(X)$. For any $b \in Im(f_1)$, we have $b = f_1(a) = f(a)$ for some $a \in K$ such that $b \in f(A) \cap L$. Since $f(A) = Im(f) = g^{-1}(U_C)$, $b \in g^{-1}(U_C) \cap L$ such that $g_1(b) = g(b) \in X$. Thus $b \in {g_1}^{-1}(X)$ and $Im(f_1) \subseteq {g_1}^{-1}(X)$. Conversely, for any $b \in {g_1}^{-1}(X)$, we have $b \in g^{-1}(U_C) \cap L$. Since $g^{-1}(U_C) = Im(f) = f(A)$, $b \in f(A) \cap L$ and $b = f(a)$ for some $a \in A$. There exists $a \in f^{-1}(f(A) \cap L) = K$, where $f_1(a) = f(a) = b$ such that $b \in Im(f_1)$. Hence ${g_1}^{-1}(X) \subseteq Im(f_1)$ and we prove that $Im(f_1) = {g_1}^{-1}(X)$.

Take $Y = h(h^{-1}(U_D) \cap M)$ and we want to show that $Im(g_1) = {h_1}^{-1}(Y)$. For any $c \in Im(g_1)$, it is clear that $c \in Im(g_1) \subseteq Im(g) = h^{-1}(U_D)$. It implies $c = g_1(b) = g(b)$ for an element $b \in L$ such that $c \in g(L) = M$. It means $c \in h^{-1}(U_D) \cap M$, so $h_1(c) = h(c) \in Y$. Hence $c \in {h_1}^{-1}(Y)$ and $Im(g_1) \subseteq {h_1}^{-1}(Y)$. Conversely, for any $c \in {h_1}^{-1}(Y)$, $c \in M = g(L)$ such that $c = g(b)$ for an element $b \in L$. It implies $c = g_1(b)$, such that $c \in Im(g_1)$. Thus, ${h_1}^{-1}(Y) \subseteq Im(g_1)$ and $Im(g_1) = {h_1}^{-1}(Y)$. $\square$

Let A, B and D be R-modules. $Hom_R(D, A)$ is the set of all R-homomorphisms from D to A. It is well known that $Hom_R(D, A)$ is a commutative group and can be viewed as a $\mathbb{Z}$-module. Now we define $\mathbb{Z}$-homomorphism

$$f_* : Hom_R(D, A) \longrightarrow Hom_R(D, B),$$

where $f_*(\alpha) = f\alpha$ for any $\alpha \in Hom_R(D, A)$. We prove a property of left quasi-exact sequences related to the $Hom_R(D, A)$.

Proposition 2.5. *Let*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \tag{2.3}$$

be a left quasi-exact sequence, which is U_C-exact in B, U_D-exact in C and let h be injective, where U_C is a submodule of C and U_D is a submodule of D. Then for any R-module M, the sequence of $\mathbb{Z}$-modules

$$0 \longrightarrow Hom_R(M, A) \xrightarrow{f_*} Hom_R(M, B) \xrightarrow{g_*} Hom_R(M, C) \xrightarrow{h_*} Hom_R(M, D) \tag{2.4}$$

is left quasi-exact, which is

HU_C-exact in $Hom_R(M, B)$ and HU_D-exact in $Hom_R(M, C)$,

where

$$HU_C = \{\rho : M \longrightarrow C \mid Im(\rho) \subseteq U_C\} \text{ and } HU_D = \{\phi : M \longrightarrow D \mid Im(\phi) \subseteq U_D\}.$$

Proof. We prove that the Sequence (2.4) is HU_C-exact in $Hom_R(M, B)$ and HU_D-exact in $Hom_R(M, C)$ by showing that f_* is injective, $Im(f_*) = g_*^{-1}(HU_C)$ and $Im(g_*) = h_*^{-1}(HU_D)$.

Take any $\theta \in Ker(f_*)$. It implies for any $x \in M$, $f\theta(x) = 0$. Since f is injective, we conclude that $\theta(x) = 0$ for any $x \in M$ such that $\theta = 0$. Hence $Ker(f_*) = 0$ and f_* is injective.

Moreover, we want to show that $Im(f_*) = g_*^{-1}(HU_C)$. Take any $\theta \in Im(f_*)$. It means $\theta = f_*(\varphi) = f\varphi$ for a homomorphism $\varphi \in Hom_R(M, A)$ such that for any $x \in M$ it hold $\theta(x) = f(\varphi(x))$ and moreover $\theta(x) \in Im(f)$. The Sequence (2.3) is U_C-exact in B, so $Im(f) = g^{-1}(U_C)$ such that $\theta(x) \in g^{-1}(U_C)$. Furthermore, $g\theta(x) \in U_C$ and $g\theta = g_*(\theta) \in HU_C$ such that $\theta \in g_*^{-1}(HU_C)$. Hence $Im(f_*) \subseteq g_*^{-1}(HU_C)$.

Conversely, take any $\gamma \in g_*^{-1}(HU_C)$. We obtain $g_*(\gamma) = g\gamma \in HU_C$, so $g\gamma(x) \in U_C$ for any $x \in M$. It implies for any $x \in M$, $\gamma(x) \in g^{-1}(U_C)$. The Sequence (2.3) is U_C-exact in B, thus $\gamma(x) \in Im(f)$. Since f is injective, we can form a homomorphism $f^{-1} : Im(f) \longrightarrow A$.

Consider now $D \xrightarrow{\gamma} Im(f) \xrightarrow{f^{-1}} A$. We define a homomorphism $h : D \longrightarrow A$ where $h = f^{-1}\gamma$. As a consequence we have

$$f_*(h) = fh = ff^{-1}\gamma = \gamma$$

such that $\gamma \in Im(f_*)$. Hence $g_*^{-1}(HU_C) \subseteq Im(f_*)$. and $Im(f_*) = g_*^{-1}(HU_C)$.

In a similar way we can also prove $Im(g_*) = h_*^{-1}(HU_D)$. Hence, the Sequence (2.4) is HU_C-exact in $Hom_R(M, B)$ and HU_D-exact in $Hom_R(M, D)$. $\square$

The converse of Proposition 2.5 is also true as we give below:

Proposition 2.6. *Let*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \qquad (2.5)$$

be a sequence of R-modules, where U_C is a submodule in C and U_D is a submodule in D. If for any R-module M, the sequence

$$0 \longrightarrow Hom_R(M, A) \xrightarrow{f_*} Hom_R(M, B) \xrightarrow{g_*} Hom_R(M, C) \xrightarrow{h_*} Hom_R(M, D) \qquad (2.6)$$

is left quasi-exact as $\mathbb{Z}$-modules which is HU_C-exact in $Hom_R(M, B)$ and HU_D-exact in $Hom_R(M, C)$, where

$$HU_C = \{\rho : M \longrightarrow C \mid Im(\rho) \subseteq U_C\} \quad dan \quad HU_D = \{\phi : M \longrightarrow D \mid Im(\phi) \subseteq U_D\}$$

then the Sequence (2.5) *is left quasi-exact.*

Proof. Take $M = Ker(f)$, such that there is an inclusion homomorphism $i \in Hom_R(M, A)$. Moreover, $f_*(i)(Ker(f)) = f(Kerf) = 0$. Since $f_*(i)$ is injective, $Ker(f) = 0$, or equivalently, f is injective.

Now we choose $M = A$. Since the Sequence (2.6) is HU_C-exact in $Hom_R(M, B)$, we have $g_*f_*(1_A) \in HU_C$. We consider that $g_*f_*(1_A) = g_*f(1_A) = gf$, so for any $x \in A$, $gf(x) \in U_C$. It implies $f(x) \in g^{-1}(U_C)$ for any $x \in A$ such that $Im(f) \subseteq g^{-1}(U_C)$. Conversely, take $M = g^{-1}(U_C)$. Then the mapping $j : M \longrightarrow B$ is an inclusion homomorphism and $gj(x) \in U_C$ for any $x \in M$. As a consequence, $g_*(j) \in HU_C$, such that $j \in g_*^{-1}(HU_C)$. The Sequence (2.6) is HU_C-exact in $Hom_R(M, B)$, so $j \in Im(f_*)$. It means $j = f_*(\alpha) = f\alpha$ for some $\alpha \in Hom_R(M, A)$. Then for any $x \in M$ hold $x = j(x) = f\alpha(x) \in Im(f)$ such that $M \subseteq Im(f)$. Hence $g^{-1}(U_C) \subseteq Im(f)$, or $Im(f) = g^{-1}(U_C)$.

Furthermore, we can prove $Im(g) \subseteq h^{-1}(U_D)$ by choosing $M = B$ and $h^{-1}(U_D) \subseteq Im(g)$ in a similar way. Hence the Sequence (2.5) is left quasi-exact. □

3 RIGHT QUASI-EXACT SEQUENCES

The important point to note here is the fact that left quasi-exact sequences can be dualized into right quasi-coexact sequences.

Definition 3.1. *A sequence* $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \longrightarrow 0$ *is called a right quasi-coexact sequence if the sequence is V_A-coexact in B, V_B-coexact in C and h is surjective, where V_A is a submodule of A and V_B is a submodule of B.*

Our next example demonstrate the existance of right quasi-coexact sequence.

Example 3.2. *The sequence $\mathbb{Z}$-modules*

$$\mathbb{Z}_{16} \xrightarrow{\pi_1} \mathbb{Z}_8 \xrightarrow{\pi_2} \mathbb{Z}_4 \xrightarrow{\pi_3} \mathbb{Z}_2 \longrightarrow 0$$

is a right quasi-coexact sequence, where π_i is a canonical projection, $i = 1, 2, 3$.

A more technical point of view of Definition 3.1 is following. A sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \longrightarrow 0$$

is right quasi-coexact if h is surjective, $f(V_A) = Ker(g)$ and $f(V_B) = Ker(h)$ for submodules V_A in A and V_B in B. Moreover, a short V-coexact

$$0 \longrightarrow B \xrightarrow{g} C \xrightarrow{h} D \longrightarrow 0$$

is a special case of a right quasi-coexact sequence, where $A = 0$. Furthermore, since every short U-exact sequence can be viewed as a short V-coexact, the short U-exact sequence $0 \longrightarrow B \xrightarrow{g} C \xrightarrow{h} D \longrightarrow 0$ is also a special condition of right quasi-coexact sequences.

Lemma 3.3. *A sequence*

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \longrightarrow 0 \tag{3.1}$$

is right quasi-coexact which is 0*-coexact in* B *if and only if the sequence*

$$0 \longrightarrow B \xrightarrow{g} C \xrightarrow{h} D \longrightarrow 0 \tag{3.2}$$

is a short V_B*-coexact sequence.*

Proof. ($\Rightarrow$) Based on the assumption and Definition 3.1, h is surjective, $f(0) = Ker(g)$, and $f(V_B) = Ker(h)$ for some submodule V_B in B. Moreover, $0 = Ker(g)$ such that g is injective. Then we can form the Sequence (3.2) where g is injective, h is surjective and $g(V_B) = Ker(h)$ for a submodule V_B in B. Hence, these sequence is short V_B-coexact.

($\Leftarrow$) Based on the assumption g is injective, h is surjective and $f(V_B) = Ker(h)$ for some submodule V_B in B. Since g is injective, $Ker(g) = 0$. For any R-homomorphism $f : A \longrightarrow B$, $f(0) = 0 = Ker(g)$. It yields the Sequence (3.1) where $f(0) = Ker(g)$, $f(V_B) = Ker(h)$ for some submodule V_B in B and h is surjective. Hence the Sequence (3.1) is right quasi-coexact which is 0-coexact in B. $\square$

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